A new complex frequency spectrum for the analysis of transmission properties in perturbed waveguides

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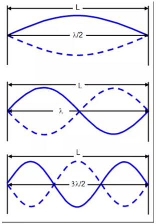
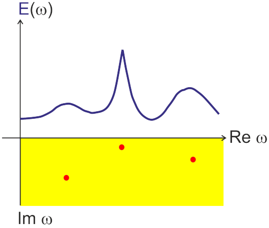
The spectral theory is classically used to study resonance phenomena:

- eigenfrequencies of a string, a closed acoustic cavity, etc...
- complex resonances of "open" cavities (with leakage)



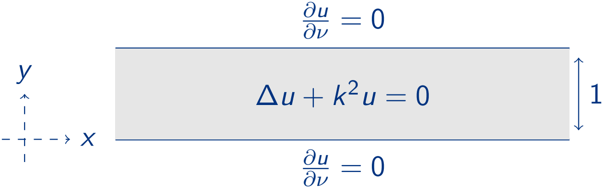


A new point of view: find similar spectral approaches to quantify the efficiency of the transmission in a waveguide.

Waveguides play an important role in optical and acoustical devices.

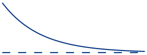
Time-harmonic scattering in waveguide

The acoustic waveguide: $\Omega = \mathbb{R} \times (0,1)$, $k = \omega / c$, $e^{-i\omega t}$

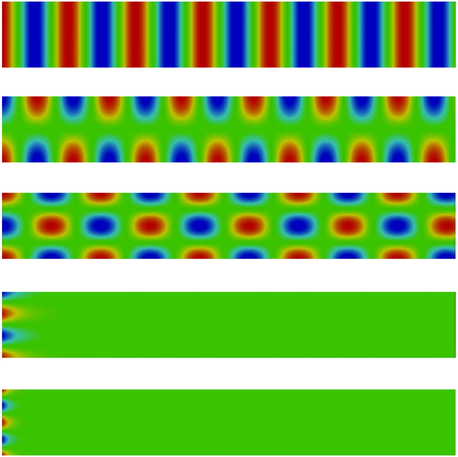


- A finite number of propagating modes for $k > n\pi$:
  $u_n^{\pm}(x,y) = \cos(n\pi y) e^{\pm i\beta_n x} \quad \beta_n = \sqrt{k^2 - n^2\pi^2}$
  ($+/-$ correspond to right/left going modes)

- An infinity of evanescent modes for $k < n\pi$:
  $u_n^{\pm}(x,y) = \cos(n\pi y) e^{\mp \gamma_n x} \quad \gamma_n = \sqrt{n^2\pi^2 - k^2}$

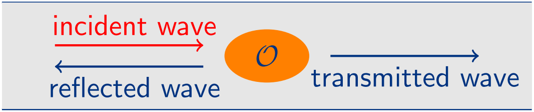


An example with 3 propagating modes:



Time-harmonic scattering in waveguide

$\mathcal{O} \subset \Omega$
$\inf(1+\rho) > 0$
$\operatorname{supp}(\rho) \subset \mathcal{O}$



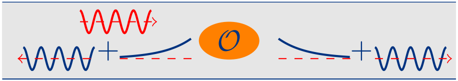
- The total field $u = u_{inc} + u_{sca}$ satisfies the equations

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

- The incident wave is a superposition of propagating modes:

$$u_{inc} = \sum_{n=0}^{N_P} a_n u_n^+$$

- The scattered field u_{sca} is outgoing:



By Fredholm analytic theory:

THEOREM

The scattering problem is well-posed except maybe for a countable set $\mathscr{T}$ of frequencies k at which trapped modes exist.

Theorem

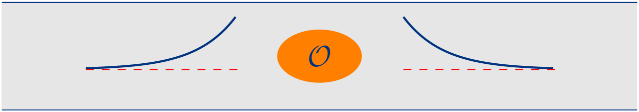
The scattering problem is well-posed except maybe for a countable set $\mathscr{T}$ of frequencies k at which trapped modes exist.

Definition

A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

such that $u \in L^2(\Omega)$.



Scattering problem and trapped modes

Theorem

The scattering problem is well-posed except maybe for a countable set $\mathscr{T}$ of frequencies k at which trapped modes exist.

Definition

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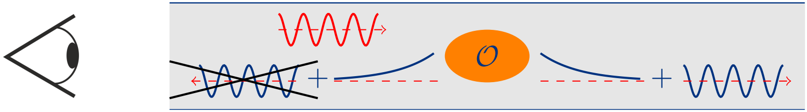
such that $u \in L^2(\Omega)$.

- There is a huge literature on trapped modes: Davies, Evans, Exner, Levitin, McIver, Nazarov, Vassiliev, ...
- Existence of trapped modes is proved in specific configurations (for instance symmetric with respect to the horizontal mid-axis) (Evans, Levitin and Vassiliev)

At particular frequencies k, it occurs that, for some u_{inc},

$$x \to -\infty \quad u_{sca} \to 0$$

We say that the obstacle $\mathcal{O}$ produces no reflection. The wave is totally transmitted. And the obstacle is invisible for an observer located far at the left-hand side.



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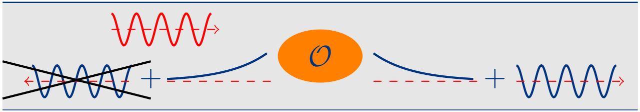
$k \in \mathscr{K}$



No-reflection

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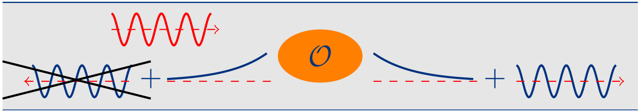
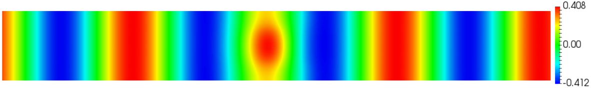


OBJECTIVE

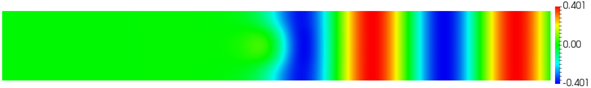
Find a way to compute directly the set $\mathscr{K}$ of no-reflection frequencies by solving an eigenvalue problem.

An illustration of no-reflection phenomenon

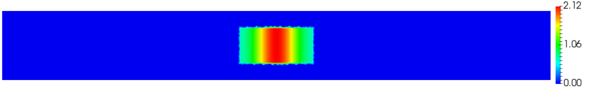
Incident field $u_{inc} = e^{ikx}$



Total field u



Scattered field u_{sca}



Perturbation ρ

The main idea

The total field u always satisfies the homogeneous equations:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega)$$

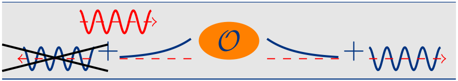
where k^2 plays the role of an eigenvalue.

Trapped modes

For $k \in \mathscr{T}$, the field of the trapped mode $u \in L^2(\Omega)$.

No-reflection

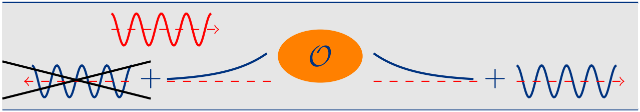
For $k \in \mathscr{K}$, the total field of the scattering problem $u \notin L^2(\Omega)$.



How to set an eigenvalue problem adapted to $\mathscr{K}$?

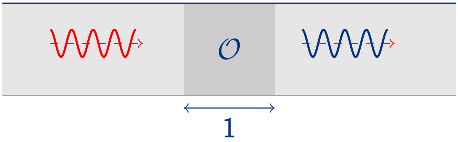
A simple and important remark

For $k \in \mathscr{K}$, the total field is ingoing at the left-hand side of $\mathcal{O}$ and outgoing at the right-hand side of $\mathcal{O}$.



The idea is to use a complex scaling (and numerically PMLs), with complex conjugate parameters at both sides of the obstacle, so that the transformed u will belong to $L^2(\Omega)$.

The 1D case



The 1D case has been studied with a spectral point of view in:

H. Hernandez-Coronado, D. Krejcirik and P. Siegl,
Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem,
Physics Letters A (2011).

Our approach allows us to extend some of their results to higher dimensions.

An additional complexity comes from the presence of evanescent modes.

Outline

1. A main tool: the complex scaling (PML)
2. Spectrum of trapped modes frequencies
3. Spectrum of no-reflection frequencies
4. Extensions and comments

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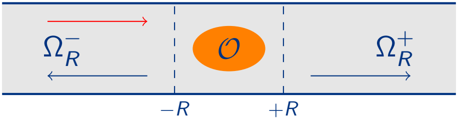
A main tool: the complex scaling

(Perfectly Matched Layers)

Perfectly Matched Layers are classically used to solve scattering problems in waveguides (Bécache et al., Kalvin, Lu et al., etc...)



We start by splitting the waveguide into three parts:

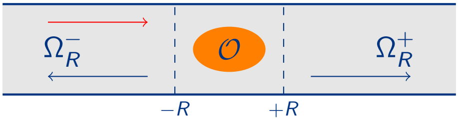
$\Omega_R = \Omega \cap \{|x| < R\}$, $\Omega_R^+ = \Omega \cap \{x > R\}$ and $\Omega_R^- = \Omega \cap \{x < -R\}$,

and we denote by:

- u the total field in Ω_R,
- u^+ the transmitted wave in Ω_R^+,
- u^- the reflected wave in Ω_R^-.

A main tool: the complex scaling

(Perfectly Matched Layers)



Reformulation of the scattering problem:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega_R) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \{|x| < R\})$$

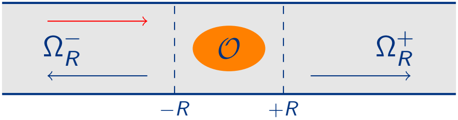
$$\Delta u^\pm + k^2 u^\pm = 0 \quad (\Omega_R^\pm) \qquad \frac{\partial u^\pm}{\partial \nu} = 0 \quad (\partial\Omega \cap \{\pm x > R\})$$

$$u = u^+ \text{ and } \frac{\partial u}{\partial x} = \frac{\partial u^+}{\partial x} \quad (x = R)$$

$$u - u_{inc} = u^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \frac{\partial u^-}{\partial x} \quad (x = -R)$$

A main tool: the complex scaling

(Perfectly Matched Layers)



Formulation with a scaling in $\Omega_R^\pm$:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega_R) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \{|x| < R\})$$

$$\Delta_\alpha u_\alpha^\pm + k^2 {u_\alpha}^\pm = 0 \quad (\Omega_R^\pm) \qquad \frac{\partial {u_\alpha}^\pm}{\partial \nu} = 0 \quad (\partial\Omega \cap \{\pm x > R\})$$

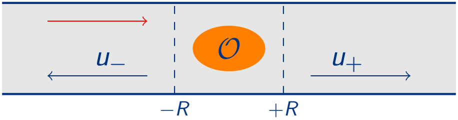
$$u = u_\alpha^+ \text{ and } \frac{\partial u}{\partial x} = \alpha \frac{\partial u_\alpha^+}{\partial x} \quad (x = R)$$

$$u - u_{inc} = u_\alpha^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \alpha \frac{\partial u_\alpha^-}{\partial x} \quad (x = -R)$$

with $u_\alpha^\pm(x,y) = u^\pm\left(\pm R + \dfrac{x \mp R}{\alpha}, y\right)$ for $(x,y) \in \Omega_R^\pm$.

A main tool: the complex scaling

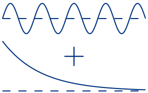
(Perfectly Matched Layers)



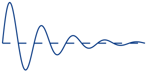
The magic idea of PMLs: take $\alpha \in \mathbb{C}$ such that $u_\alpha^\pm \in L^2(\Omega_R^\pm)$.

If $\alpha = e^{-i\theta}$ with $0 < \theta < \pi/2$, propagating modes become evanescent:

$$
\begin{aligned}
u^+(x,y) = & \sum_{n \le N_P} a_n \cos(n\pi y) e^{i\sqrt{k^2 - n^2\pi^2}(x-R)} \\
+ & \sum_{n > N_P} a_n \cos(n\pi y) e^{-\sqrt{n^2\pi^2 - k^2}(x-R)}
\end{aligned}
$$



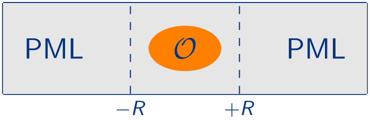
$$
\begin{aligned}
u_\alpha^+(x,y) = & \sum_{n \le N_P} a_n \cos(n\pi y) e^{\frac{i\sqrt{k^2 - n^2\pi^2}}{\alpha}(x-R)} \\
+ & \sum_{n > N_P} a_n \cos(n\pi y) e^{-\frac{\sqrt{n^2\pi^2 - k^2}}{\alpha}(x-R)}
\end{aligned}
$$



and the same for u_α^- with the same α.

A main tool: the complex scaling

(Perfectly Matched Layers)



Final PML formulation:

$$\Delta u + k^2(1+\rho)u = 0 \quad (\Omega_R) \qquad \frac{\partial u}{\partial \nu} = 0 \quad (\partial\Omega \cap \cap\{|x| < R\}$$

$$\Delta_\alpha u_\alpha^\pm + k^2 u_\alpha{}^\pm = 0 \quad (\Omega_R^\pm) \qquad \frac{\partial u_\alpha{}^\pm}{\partial \nu} = 0 \quad (\partial\Omega \cap \{\pm x > R\})$$

$$u = u_\alpha^+ \text{ and } \frac{\partial u}{\partial x} = \alpha \frac{\partial u_\alpha^+}{\partial x} \quad (x = R)$$

$$u - u_{inc} = u_\alpha^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \alpha \frac{\partial u_\alpha^-}{\partial x} \quad (x = -R)$$

where $\Delta_\alpha = e^{-2i\theta}\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ and $u_\alpha^\pm \in L^2(\Omega_R^\pm)$.

Outline

Definition

A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of

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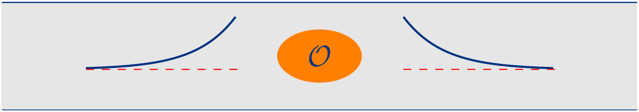
such that $u \in L^2(\Omega)$.



The spectral problem for trapped modes

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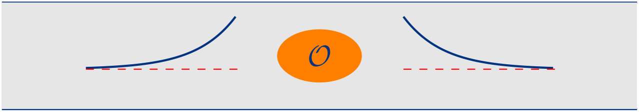

Let us consider the following unbounded operator of $L^2(\Omega)$:

$$D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \qquad Au = -\frac{1}{1+\rho}\Delta u$$

$$\Delta u + k^2(1+\rho)u = 0 \iff Au = k^2 u$$

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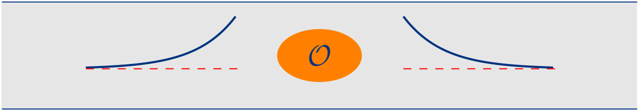

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The trapped modes ($k \in \mathscr{T}$) correspond to real eigenvalues k^2 of A.

A is an unbounded operator with domain $D(A) \subset H$ (H Hilbert space)

RESOLVENT SET AND SPECTRUM

$\rho(A) = \{\lambda \in \mathbb{C}; A - \lambda I$ is bijective from $D(A)$ to $H\}$ and $\sigma(A) = \mathbb{C} \backslash \rho(A)$

The spectrum $\sigma(A)$ contains the eigenvalues but not only....

ESSENTIAL SPECTRUM

If $u_n \in D(A)$, $\|u_n\| = 1$, $u_n \rightharpoonup 0$ and $\|Au_n - \lambda u_n\| \to 0$ (Weyl sequence), we say that $\lambda \in \sigma_{ess}(A)$.

The essential spectrum $\sigma_{ess}(A)$ is stable under compact perturbations.

DISCRETE SPECTRUM

$\sigma_{disc}(A)$ is the set of isolated eigenvalues with finite multiplicity.

If A is self-adjoint, $\sigma(A) = \sigma_{ess}(A) \cup \sigma_{disc}(A) \subset \mathbb{R}$.

The spectral problem for trapped modes

Trapped modes ($k \in \mathscr{T}$) correspond to real eigenvalues k^2 of

$$Au = -\frac{1}{1+\rho}\Delta u \quad \text{with } D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

For the scalar product of $L^2(\Omega)$ with weight $1+\rho$:

The spectral problem for trapped modes

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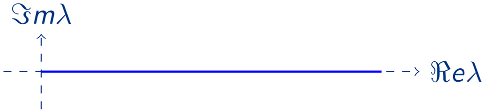
For the scalar product of $L^2(\Omega)$ with weight $1+\rho$:

Spectral features of A

- A is a positive self-adjoint operator.
- $\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+$ and $\sigma_{disc}(A) = \emptyset$



The spectral problem for trapped modes

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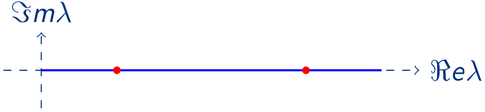
For the scalar product of $L^2(\Omega)$ with weight $1+\rho$:

Spectral features of A

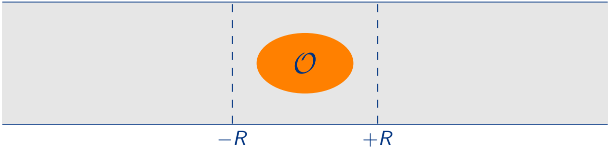
- A is a positive self-adjoint operator.
- $\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+$ and $\sigma_{disc}(A) = \emptyset$
- Trapped modes are embedded eigenvalues of A !



Solution: the complex scaling (Aguilar, Balslev, Combes, Simon 70)

Let us consider now the following unbounded operator:

$$\begin{aligned} D(A_\alpha) &= \{u \in L^2(\Omega); A_\alpha u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\ A_\alpha u &= -\frac{1}{1+\rho(x,y)}\left(\alpha(x)\frac{\partial}{\partial x}\left(\alpha(x)\frac{\partial u}{\partial x}\right) + \frac{\partial^2 u}{\partial y^2}\right) \end{aligned}$$

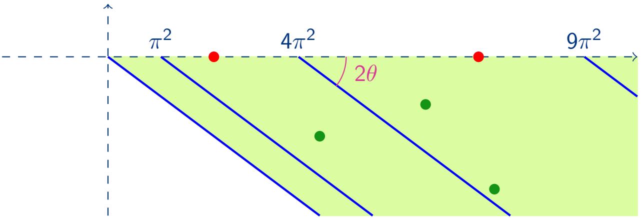


where $\alpha(x) = e^{-i\theta}$ $\alpha(x) = 1$ $\alpha(x) = e^{-i\theta}$

Spectral features of A_α

- A_α is a non self-adjoint operator.
- $\sigma_{ess}(A_\alpha) = \cup_{n\geq 0}\{n^2\pi^2 + e^{-2i\theta}t^2; t \in \mathbb{R}\}$
- $\sigma(A_\alpha) = \sigma_{ess}(A_\alpha) \,\dot\cup\, \sigma_{disc}(A_\alpha)$
- $\sigma(A_\alpha) \subset \{z \in \mathbb{C}; -2\theta < \arg(z) \leq 0\}$

(see Kalvin, Kim and Pasciak)



Some elements of proof

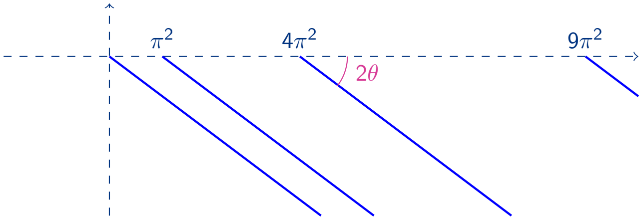
Proof of the second item:

$$\begin{aligned}\sigma_{ess}(A_\alpha) &= \sigma_{ess}(-\Delta_\theta) \\ &= \bigcup_{n\geq 0} \sigma_{ess}(-\Delta_\theta^{(n)}) \\ &= \bigcup_{n\geq 0} \{n^2\pi^2 + e^{-2i\theta}t^2; t \in \mathbb{R}\}\end{aligned}$$

$$\Delta_\theta = e^{-2i\theta}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Delta_\theta^{(n)} = e^{-2i\theta}\frac{\partial^2}{\partial x^2} + n^2\pi^2$$

Essential spectrum of A_α:



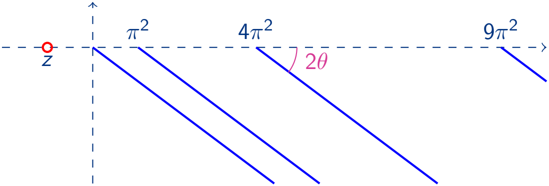
Some elements of proof

Proof of the third item: $\sigma(A_\alpha) = \sigma_{ess}(A_\alpha) \cup \sigma_{disc}(A_\alpha)$

The result follows from analytic Fredholm theorem because:

1. $U = \mathbb{C} \backslash \sigma_{ess}(A_\alpha)$ is a connected set.
2. There is a point $z \in U$ such that $A_\alpha - z$ is invertible (coerciveness).

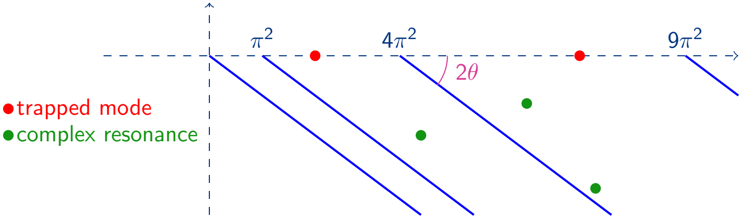
(See D.E. Edmunds and W.D. Evans, Spectral theory and differential operators.)



Discrete spectrum of A_α

- Trapped modes correspond to discrete real eigenvalues of A_α !
- Other eigenvalues correspond to complex resonances, with a field u exponentially growing at infinity.

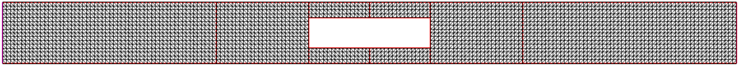
Spectrum of A_α:



Numerical illustration

The numerical results have been obtained by a finite element discretization with FreeFem++.

Here the scatterer is a non-penetrable rectangular obstacle in the middle of the waveguide:



We put PMLs in the magenta parts:

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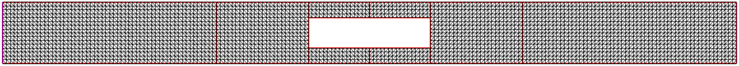
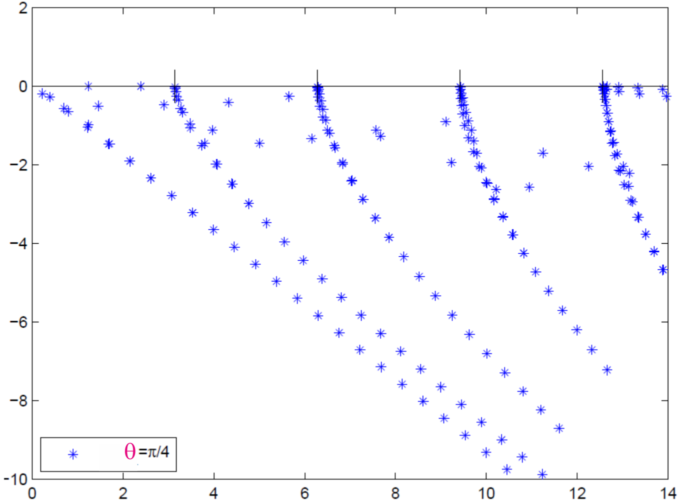


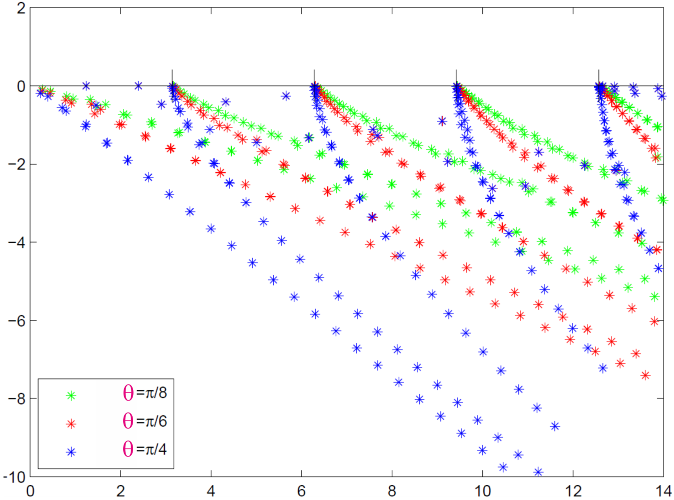
We put PMLs in the magenta parts:

In the next slides, we represent the square-root of the spectrum, which corresponds to k values.

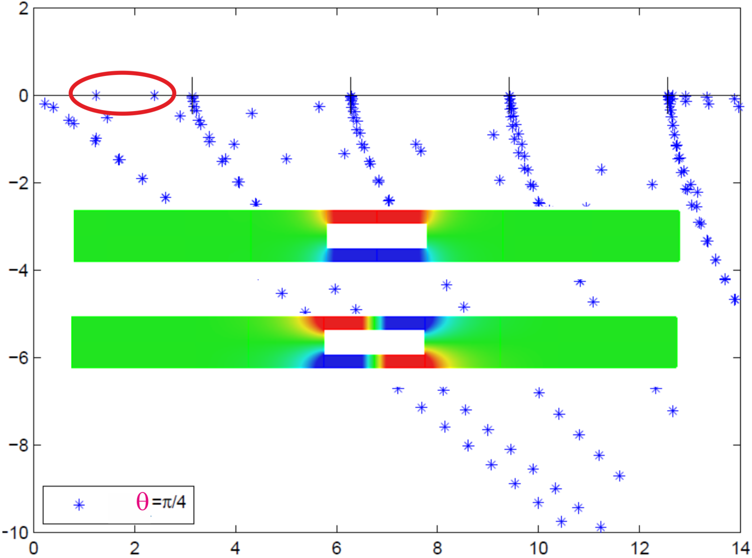
Numerical illustration



Numerical illustration



There are two trapped modes:



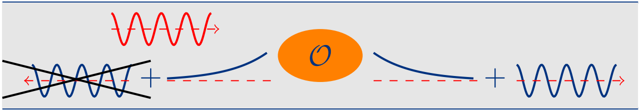
Outline

A NEW COMPLEX SPECTRUM LINKED TO $\mathscr{K}$

WITH "CONJUGATE" PMLS

A SIMPLE AND IMPORTANT REMARK

For $k \in \mathscr{K}$, the total field is ingoing at the left-hand side of $\mathcal{O}$ and outgoing at the right-hand side of $\mathcal{O}$.



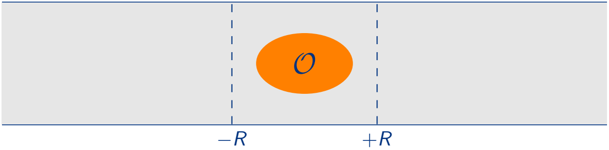
The idea is to use a complex scaling (and numerically PMLs), with complex conjugate parameters at both sides of the obstacle, so that the transformed total field u will belong to $L^2(\Omega)$.

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

Let us consider now the following unbounded operator:

$$
\begin{aligned}
D(A_{\tilde{\alpha}}) &= \{u \in L^2(\Omega); A_{\tilde{\alpha}} u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\} \\
A_{\tilde{\alpha}} u &= -\frac{1}{1+\rho(x,y)} \left(\tilde{\alpha}(x) \frac{\partial}{\partial x} \left(\tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)
\end{aligned}
$$



$\tilde{\alpha}(x) = e^{i\theta}$ $\quad$ $\tilde{\alpha}(x) = 1$ $\quad$ $\tilde{\alpha}(x) = e^{-i\theta}$

A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

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\end{aligned}
$$

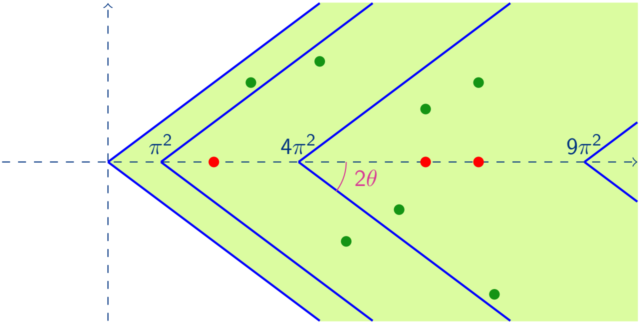
Spectral features of $A_{\tilde{\alpha}}$

- $A_{\tilde{\alpha}}$ is a non self-adjoint operator.
- $\sigma_{ess}(A_{\tilde{\alpha}}) = \bigcup_{n\geq 0}\{n^2\pi^2 + e^{2i\theta} t^2; t \in \mathbb{R}\} \cup \{n^2\pi^2 + e^{-2i\theta} t^2; t \in \mathbb{R}\}$
- $\sigma_{disc}(A_{\tilde{\alpha}}) \subset \{z \in \mathbb{C}; -2\theta < \arg(z) < 2\theta\}$

A NEW COMPLEX SPECTRUM LINKED TO $\mathscr{K}$

WITH "CONJUGATE" PMLS

Typical expected spectrum of $A_{\tilde{\alpha}}$:



SPECTRAL FEATURES OF $A_{\tilde{\alpha}}$

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A new complex spectrum linked to $\mathscr{K}$

with "conjugate" PMLs

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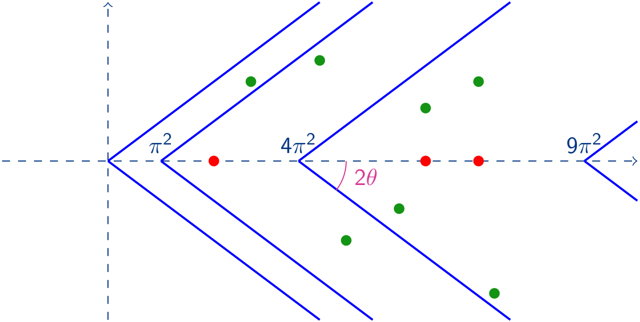


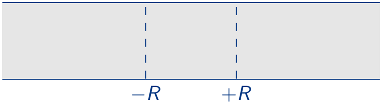
Difficulty: $\mathbb{C} \backslash \sigma_{ess}(A_{\tilde{\alpha}})$ is not a connected set.

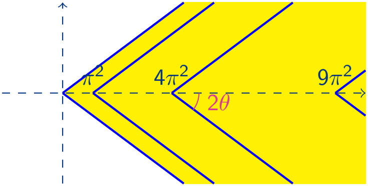
Conjecture

$$\sigma(A_{\tilde{\alpha}}) = \sigma_{ess}(A_{\tilde{\alpha}}) \cup \sigma_{disc}(A_{\tilde{\alpha}}) \text{ if } \rho \neq 0$$

Pathological cases

In the unperturbed case ($\rho = 0$):





All k^2 in the yellow zone are eigenvalues of $A_{\tilde{\alpha}}$!

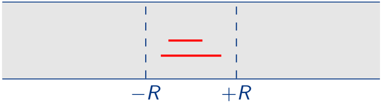
Proof: Use the strechted plane wave as an eigenvector:

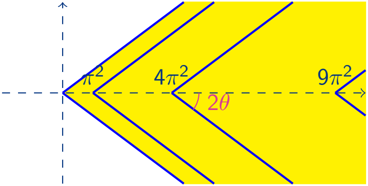
$$A_{\tilde{\alpha}} u = k^2 u$$

for $u(x, y) = \begin{cases} e^{ik(-R+(x+R)e^{-i\theta})} & \text{if} \quad x < -R \\ e^{ikx} & \text{if} \quad -R < x < R \\ e^{ik(R+(x-R)e^{i\theta})} & \text{if} \quad R < x \end{cases}$

Pathological cases

And the same result holds with horizontal cracks !





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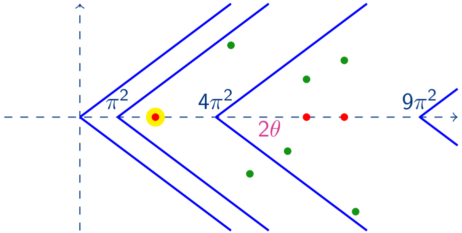


For real eigenvalues, the eigenmode is such that

u is ingoing $\mathcal{O}$ u is outgoing

Link between the discrete spectrum and $\mathscr{K}$

For $k^2 \in \sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R}$, the eigenmode is such that:

There are two cases:

- Either u on the left-hand side contains a propagating part and it is a case of no-reflection: $k \in \mathscr{K}$.
- Either u is evanescent on both sides and k is associated to a trapped mode: $k \in \mathscr{T}$.

Theorem

$$\sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R} = \{k^2 \in \mathbb{R}; k \in \mathscr{K} \cup \mathscr{T}\}$$

$\mathcal{PT}$-SYMMETRY (SPACE-TIME REFLECTION SYMMETRY)

Remember that:

$$A_{\tilde{\alpha}} u = -\frac{1}{1+\rho(x,y)} \left(\tilde{\alpha}(x) \frac{\partial}{\partial x} \left(\tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)$$

and that

$$\tilde{\alpha}(-x) = \overline{\tilde{\alpha}(x)}$$

CONSEQUENCE

If the obstacle is symmetric in x:

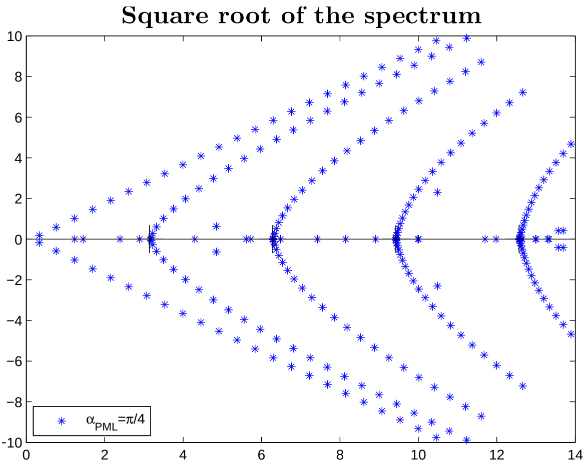
$$\rho(-x, y) = \rho(x, y)$$

$A_{\tilde{\alpha}}$ is $\mathcal{PT}$-symmetric and its spectrum is stable by complex conjugation:

$$\sigma(A_{\tilde{\alpha}}) = \overline{\sigma(A_{\tilde{\alpha}})}$$

Numerical illustration

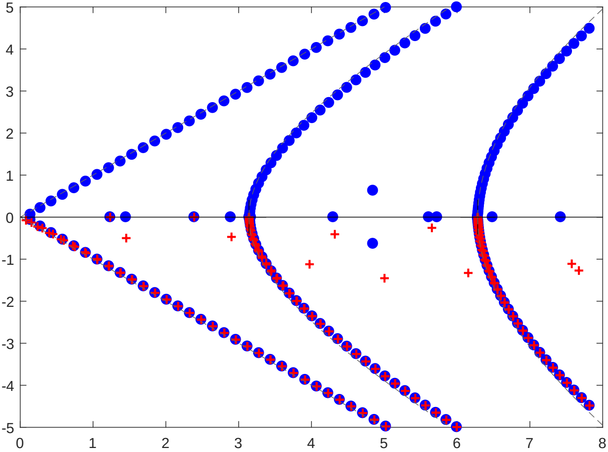
for a rectangular symmetric cavity



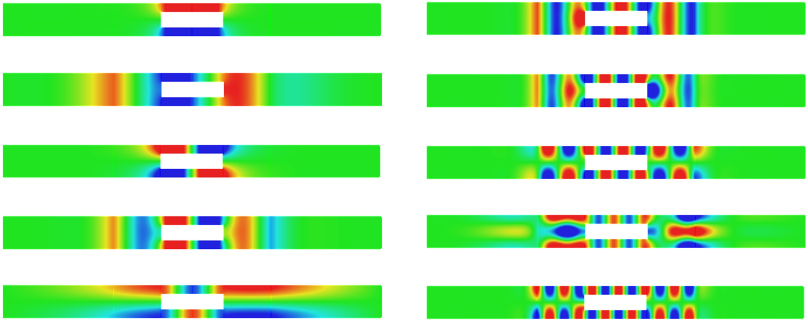
- The spectrum is symmetric w.r.t. the real axis ($\mathcal{PT}$-symmetry) .
- There are much more real eigenvalues than for trapped modes.

Numerical illustration

for a rectangular symmetric cavity



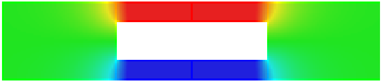
Red: classical PMLs
Blue: conjugate PMLs



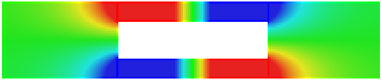
This is a representation of the computed modes for the 10 first real eigenvalues and in the whole computational domain (including PMLs).

Validation

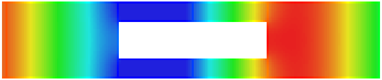
Let us focus on the eigenmodes such that $0 < k < \pi$:



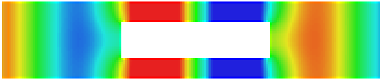
First trapped mode:
$k = 1.2355\cdots$



Second trapped mode:
$k = 2.3897\cdots$



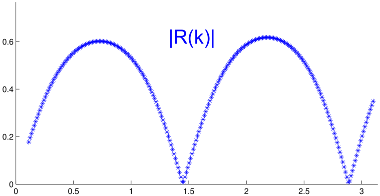
First no-reflection mode:
$k = 1.4513\cdots$

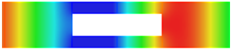


Second no-reflection mode:
$k = 2.8896\cdots$

Validation

To validate this result, we compute the amplitude of the reflected plane wave for $0 < k < \pi$:





First no-reflection mode: $k = 1.4513\cdots$

Second no-reflection mode: $k = 2.8896\cdots$

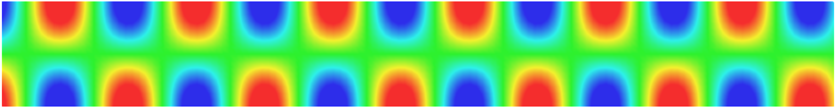
There is a perfect agreement!

No-reflection mode in the time-domain

Below we represent $\Re e(u(x,y)e^{-i\omega t})$ with u...

...a no-reflection mode:

with the corresponding incident propagating mode:



We observe no reflection but a phase shift in the transmitted wave.

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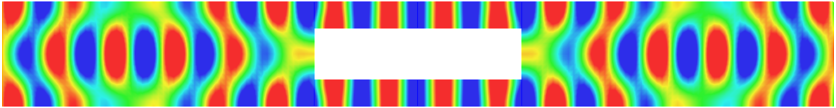

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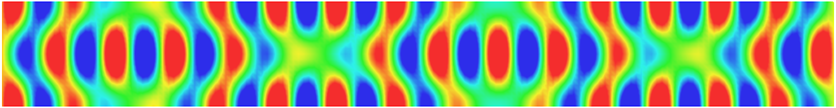
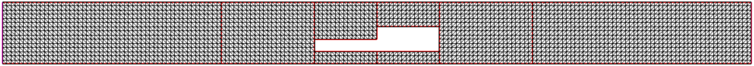


We observe no reflection but a phase shift in the transmitted wave.

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

Here the scatterer is a not symmetric in x, and neither in y:

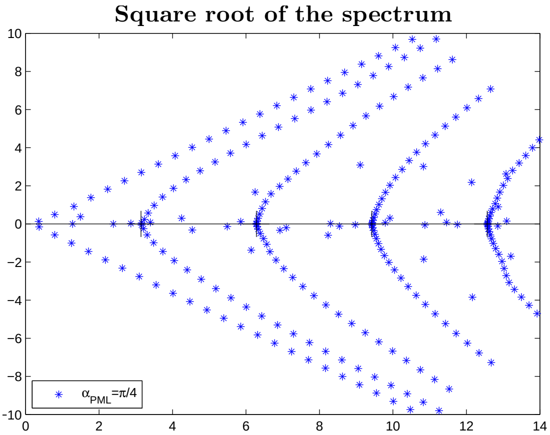


We expect:

- No trapped modes
- No invariance of the spectrum by complex conjugation

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

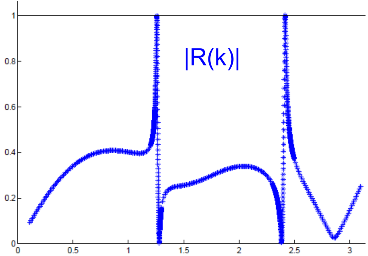


- The spectrum is no longer symmetric w.r.t. the real axis.
- There are several eigenvalues near the real axis.

Numerical illustration

in a non $\mathcal{PT}$-symmetric case

Again results can be validated by computing $R(k)$ for $0 < k < \pi$:



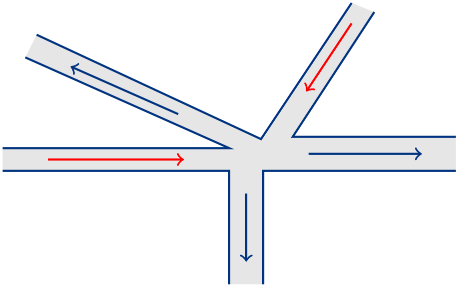


$k = 1.2803 + 0.0003i$ $k = 2.3868 + 0.0004i$ $k = 2.8650 + 0.0241i$

Complex eigenvalues also contain useful information about almost no-reflection.

Outline

Multiport waveguides junction



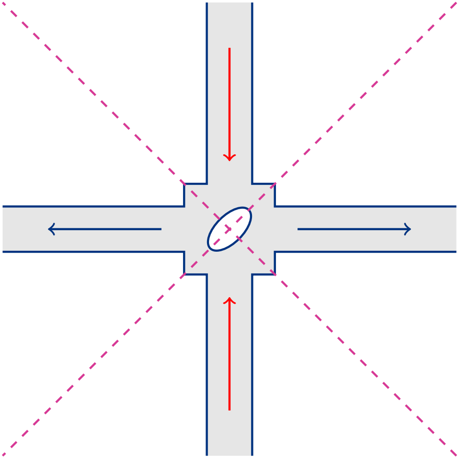
OBJECTIVE

Find (k, u) such that u is ingoing in some ports and outgoing in the others.

For an N-ports junction, there are 2^{N-1} such problems and corresponding spectra.

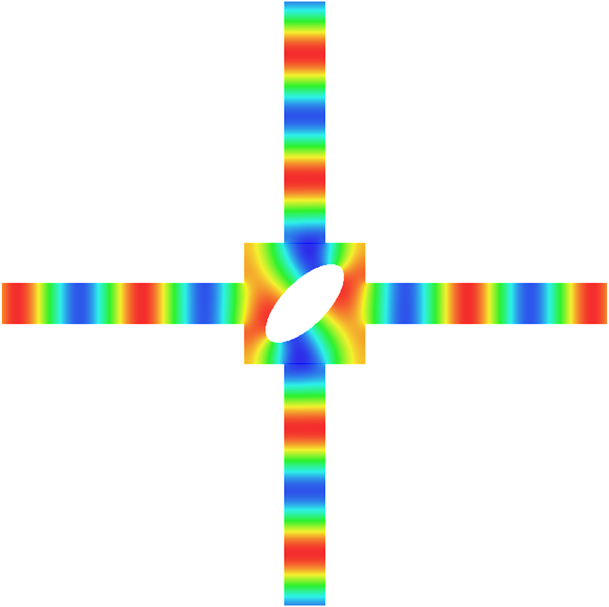
Multiport waveguides junction

This is a bar-bar example of such problem:



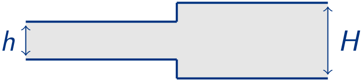
There are two axes of $\mathcal{PT}$-symmetry!

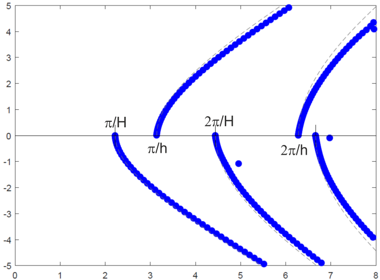
Multiport waveguides junction



Junction of Dirichlet waveguides

An interesting configuration is the junction of 2 different Dirichlet waveguides.





Consequences

- Now $\mathbb{C}\backslash\sigma_{ess}(A_{\tilde{\alpha}})$ is a connected set!
- Our "new" eigenvalues correspond in fact to classical complex resonances in non-classical sheets of the Riemannn surface......

Conclusion

There is still a lot of work to do !

- Clarify the link between our new spectrum and classical resonance frequencies.
- Prove the existence of no-reflection frequencies ($\mathscr{K} \neq \emptyset$), at least in $\mathcal{PT}$-symmetric cases.
- Justify the numerics (absence of spectral pollution).
- Find similar spectral approaches for other phenomena in waveguides (perfect invisibility, total reflection, modal conversion, etc...)
- ...

These results have been published in:

Trapped modes and reflectionless modes as eigenfunctions of the same spectral problem, Anne-Sophie Bonnet-BenDhia, Lucas Chesnel and Vincent Pagneux, Proceedings of the Royal Society A, 2018.


Thank You