A NEW COMPLEX FREQUENCY SPECTRUM FOR THE ANALYSIS OF TRANSMISSION PROPERTIES IN PERTURBED WAVEGUIDES

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The spectral theory is classically used to study resonance phenomena:

- **eigenfrequencies** of a string, a closed acoustic cavity, etc...
- **complex resonances** of “open” cavities (with leakage)

A new point of view: find similar spectral approaches to quantify the efficiency of the transmission in a waveguide.

Waveguides play an important role in optical and acoustical devices.
The acoustic waveguide: \( \Omega = \mathbb{R} \times (0, 1) \), \( k = \omega / c \), \( e^{-i\omega t} \)

\[ \Delta u + k^2 u = 0 \]

\[ \frac{\partial u}{\partial \nu} = 0 \]

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- A finite number of propagating modes for \( k > n\pi \):
  \[ u_n^\pm(x, y) = \cos(n\pi y)e^{\pm i\beta_n x} \quad \beta_n = \sqrt{k^2 - n^2\pi^2} \]
  (+/- correspond to right/left going modes)

- An infinity of evanescent modes for \( k < n\pi \):
  \[ u_n^\pm(x, y) = \cos(n\pi y)e^{\mp i\gamma_n x} \quad \gamma_n = \sqrt{n^2\pi^2 - k^2} \]
An example with 3 propagating modes:
The total field $u = u_{inc} + u_{sca}$ satisfies the equations

$$\Delta u + k^2(1 + \rho)u = 0 \quad (\Omega) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega)$$

The incident wave is a superposition of propagating modes:

$$u_{inc} = \sum_{n=0}^{N_P} a_n u_n^+$$

The scattered field $u_{sca}$ is outgoing:
By Fredholm analytic theory:

**THEOREM**

The scattering problem is well-posed except maybe for a countable set $\mathcal{I}$ of frequencies $k$ at which trapped modes exist.
Theorem
The scattering problem is well-posed except maybe for a countable set $\mathcal{T}$ of frequencies $k$ at which trapped modes exist.

Definition
A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of
\[
\Delta u + k^2 (1 + \rho) u = 0 \quad (\Omega) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega)
\]
such that $u \in L^2(\Omega)$. 
The scattering problem is well-posed except maybe for a countable set $\mathcal{I}$ of frequencies $k$ at which trapped modes exist.

A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of

$$
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$$

such that $u \in L^2(\Omega)$.

- There is a huge literature on trapped modes: Davies, Evans, Exner, Levitin, McIver, Nazarov, Vassiliev, ...
- Existence of trapped modes is proved in specific configurations (for instance symmetric with respect to the horizontal mid-axis) (Evans, Levitin and Vassiliev)
At particular frequencies $k$, it occurs that, for some $u_{inc}$,

$$x \to -\infty \quad u_{sca} \to 0$$

We say that the obstacle $\mathcal{O}$ produces no reflection. The wave is totally transmitted. And the obstacle is invisible for an observer located far at the left-hand side.
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$\mathcal{H}$

**OBJECTIVE**

Find a way to compute directly the set $\mathcal{H}$ of no-reflection frequencies by solving an eigenvalue problem.
AN ILLUSTRATION OF NO-REFLECTION PHENOMENON

Incident field $u_{inc} = e^{ikx}$

Total field $u$

Scattered field $u_{sca}$

Perturbation $\rho$
The main idea

The total field $u$ always satisfies the homogeneous equations:

$$\Delta u + k^2(1 + \rho)u = 0 \quad (\Omega) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega)$$

where $k^2$ plays the role of an eigenvalue.

**Trapped modes**

For $k \in \mathcal{T}$, the field of the trapped mode $u \in L^2(\Omega)$.

**No-reflection**

For $k \in \mathcal{K}$, the total field of the scattering problem $u \notin L^2(\Omega)$.

How to set an eigenvalue problem adapted to $\mathcal{K}$?
The main idea

A simple and important remark

For \( k \in \mathcal{K} \), the total field is \textbf{ingoing} at the left-hand side of \( \mathcal{O} \) and \textbf{outgoing} at the right-hand side of \( \mathcal{O} \).

The idea is to use a \textbf{complex scaling} (and numerically \textbf{PMLs}), with complex \textbf{conjugate} parameters at both sides of the obstacle, so that the transformed \( u \) will belong to \( L^2(\Omega) \).
The 1D case has been studied with a spectral point of view in:


Our approach allows us to extend some of their results to higher dimensions.

An additional complexity comes from the presence of evanescent modes.
Outline

1. A main tool: the complex scaling (PML)
2. Spectrum of trapped modes frequencies
3. Spectrum of no-reflection frequencies
4. Extensions and comments
Outline

1 A main tool: the complex scaling (PML)

2 Spectrum of trapped modes frequencies

3 Spectrum of no-reflection frequencies

4 Extensions and comments
**A main tool: the complex scaling**

*(Perfectly Matched Layers)*

Perfectly Matched Layers are classically used to solve scattering problems in waveguides (Bécache et al., Kalvin, Lu et al., etc...)

We start by splitting the waveguide into three parts:

\[ \Omega_R = \Omega \cap \{|x| < R\}, \Omega_R^+ = \Omega \cap \{x > R\} \text{ and } \Omega_R^- = \Omega \cap \{x < -R\}, \]

and we denote by:

- \( u \) the total field in \( \Omega_R \),
- \( u^+ \) the transmitted wave in \( \Omega_R^+ \),
- \( u^- \) the reflected wave in \( \Omega_R^- \).
A main tool: the complex scaling
(Perfectly Matched Layers)

Reformulation of the scattering problem:

\[ \Delta u + k^2(1 + \rho)u = 0 \quad (\Omega_R) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega \cap \{|x| < R\}) \]

\[ \Delta u^\pm + k^2 u^\pm = 0 \quad (\Omega_R^\pm) \quad \frac{\partial u^\pm}{\partial \nu} = 0 \quad (\partial \Omega \cap \{\pm x > R\}) \]

\[ u = u^+ \text{ and } \frac{\partial u}{\partial x} = \frac{\partial u^+}{\partial x} \quad (x = R) \]

\[ u - u_{inc} = u^- \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \frac{\partial u^-}{\partial x} \quad (x = -R) \]
**A main tool: the complex scaling**
*(Perfectly Matched Layers)*

Formulation with a scaling in $\Omega^\pm_R$:

\[ \Delta u + k^2(1 + \rho)u = 0 \quad (\Omega_R) \]
\[ \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega \cap \{|x| < R\}) \]
\[ \Delta_{\alpha} u^\pm_{\alpha} + k^2 u^\pm_{\alpha} = 0 \quad (\Omega^\pm_R) \]
\[ \frac{\partial u^\pm_{\alpha}}{\partial \nu} = 0 \quad (\partial \Omega \cap \{\pm x > R\}) \]

\[ u = u^+_\alpha \text{ and } \frac{\partial u}{\partial x} = \alpha \frac{\partial u^+_\alpha}{\partial x} \quad (x = R) \]

\[ u - u_{inc} = u^-_{\alpha} \text{ and } \frac{\partial}{\partial x}(u - u_{inc}) = \alpha \frac{\partial u^-_{\alpha}}{\partial x} \quad (x = -R) \]

With $u^\pm_{\alpha}(x, y) = u^\pm \left(\pm R + \frac{x \mp R}{\alpha}, y\right)$ for $(x, y) \in \Omega^\pm_R$. 
A main tool: the complex scaling

(Perfectly Matched Layers)

The magic idea of PMLs: take $\alpha \in \mathbb{C}$ such that $u^\pm_{\alpha} \in L^2(\Omega^\pm_R)$.

If $\alpha = e^{-i\theta}$ with $0 < \theta < \pi/2$, propagating modes become evanescent:

$$u^+(x, y) = \sum_{n \leq N_P} a_n \cos(n\pi y) e^{i\sqrt{k^2 - n^2\pi^2}(x-R)} + \sum_{n > N_P} a_n \cos(n\pi y) e^{-\sqrt{n^2\pi^2-k^2}(x-R)}$$

$$u^\pm_{\alpha}(x, y) = \sum_{n \leq N_P} a_n \cos(n\pi y) e^{\frac{i\sqrt{k^2 - n^2\pi^2}}{\alpha}(x-R)} + \sum_{n > N_P} a_n \cos(n\pi y) e^{-\frac{\sqrt{n^2\pi^2-k^2}}{\alpha}(x-R)}$$

and the same for $u^-_{\alpha}$ with the same $\alpha$. 
**A main tool: the complex scaling**  
* (Perfectly Matched Layers)  

**Final PML formulation:**

\[
\Delta u + k^2 (1 + \rho) u = 0 \quad (\Omega_R) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega \cap \cap \{|x| < R\})
\]

\[
\Delta_\alpha u_\alpha^\pm + k^2 u_\alpha^\pm = 0 \quad (\Omega_R^\pm) \quad \frac{\partial u_\alpha^\pm}{\partial \nu} = 0 \quad (\partial \Omega \cap \{\pm x > R\})
\]

\[\begin{align*}
  u &= u_\alpha^+ \quad \text{and} \quad \frac{\partial u}{\partial x} = \alpha \frac{\partial u_\alpha^+}{\partial x} \quad (x = R) \\
  u - u_{\text{inc}} &= u_\alpha^- \quad \text{and} \quad \frac{\partial}{\partial x} (u - u_{\text{inc}}) = \alpha \frac{\partial u_\alpha^-}{\partial x} \quad (x = -R)
\end{align*}\]

where \( \Delta_\alpha = e^{-2i\theta} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and \( u_\alpha^\pm \in L^2(\Omega_R^\pm) \).
Outline

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2. Spectrum of trapped modes frequencies

3. Spectrum of no-reflection frequencies

4. Extensions and comments
The spectral problem for trapped modes

**Definition**

A trapped mode of the perturbed waveguide is a solution $u \neq 0$ of

$$
\Delta u + k^2(1 + \rho)u = 0 \quad (\Omega) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega)
$$

such that $u \in L^2(\Omega)$. 

\[\Box\]
The spectral problem for trapped modes

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\Delta u + k^2(1 + \rho)u = 0 \quad (\Omega) \quad \frac{\partial u}{\partial \nu} = 0 \quad (\partial \Omega)
\]

such that \( u \in L^2(\Omega) \).

Let us consider the following unbounded operator of \( L^2(\Omega) \):

\[
D(A) = \{ u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \} \quad Au = -\frac{1}{1 + \rho} \Delta u
\]

\[
\Delta u + k^2(1 + \rho)u = 0 \iff Au = k^2 u
\]
The spectral problem for trapped modes

**Definition**

A trapped mode of the perturbed waveguide is a solution \( u \neq 0 \) of

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\]

The trapped modes \((k \in \mathcal{T})\) correspond to real eigenvalues \( k^2 \) of \( A \).
A is an unbounded operator with domain $D(A) \subset H$ ($H$ Hilbert space)

**Resolvent set and spectrum**

\[ \rho(A) = \{ \lambda \in \mathbb{C}; A - \lambda I \text{ is bijective from } D(A) \text{ to } H \} \text{ and } \sigma(A) = \mathbb{C} \setminus \rho(A) \]

The spectrum $\sigma(A)$ contains the eigenvalues but not only....

**Essential spectrum**

If $u_n \in D(A), \|u_n\| = 1$, $u_n \rightharpoondown 0$ and $\|Au_n - \lambda u_n\| \rightarrow 0$ (Weyl sequence), we say that $\lambda \in \sigma_{\text{ess}}(A)$.

The essential spectrum $\sigma_{\text{ess}}(A)$ is stable under compact perturbations.

**Discrete spectrum**

$\sigma_{\text{disc}}(A)$ is the set of isolated eigenvalues with finite multiplicity.

If $A$ is self-adjoint, $\sigma(A) = \sigma_{\text{ess}}(A) \cup \sigma_{\text{disc}}(A) \subset \mathbb{R}$. 
Trapped modes \((k \in \mathcal{T})\) correspond to real eigenvalues \(k^2\) of

\[
Au = -\frac{1}{1 + \rho} \Delta u \quad \text{with } D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega\}
\]

For the scalar product of \(L^2(\Omega)\) with weight \(1 + \rho\):

\[
\text{Spectral features of } A
\]

\[
\sigma(A) = \sigma_{\text{ess}}(A) = \mathbb{R}^+ \text{ and } \sigma_{\text{disc}}(A) = \emptyset
\]
Trapped modes \((k \in \mathcal{T})\) correspond to real eigenvalues \(k^2\) of

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Au = -\frac{1}{1 + \rho} \Delta u \quad \text{with } D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega\}
\]

For the scalar product of \(L^2(\Omega)\) with weight \(1 + \rho\):

**Spectral features of \(A\)**

- \(A\) is a positive self-adjoint operator.
- \(\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+\) and \(\sigma_{disc}(A) = \emptyset\)
Trapped modes \((k \in \mathcal{T})\) correspond to real eigenvalues \(k^2\) of

\[
Au = -\frac{1}{1 + \rho} \Delta u \quad \text{with } D(A) = \{u \in H^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega\}
\]

For the scalar product of \(L^2(\Omega)\) with weight \(1 + \rho\):

**Spectral features of** \(A\)
- \(A\) is a positive self-adjoint operator.
- \(\sigma(A) = \sigma_{ess}(A) = \mathbb{R}^+\) and \(\sigma_{disc}(A) = \emptyset\)
- Trapped modes are embedded eigenvalues of \(A\)!

\[\Im m \lambda \uparrow \quad - \rightarrow \Re e \lambda\]

**Solution:** the complex scaling (Aguilar, Balslev, Combes, Simon 70)
Let us consider now the following unbounded operator:

\[ D(A_\alpha) = \{ u \in L^2(\Omega); A_\alpha u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \} \]

\[ A_\alpha u = -\frac{1}{1+\rho(x,y)} \left( \alpha(x) \frac{\partial}{\partial x} \left( \alpha(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right) \]

where \( \alpha(x) = e^{-i\theta} \), \( \alpha(x) = 1 \), \( \alpha(x) = e^{-i\theta} \)
**Spectral features of $A_\alpha$**

- $A_\alpha$ is a **non self-adjoint** operator.
- $\sigma_{ess}(A_\alpha) = \bigcup_{n \geq 0} \{ n^2 \pi^2 + e^{-2i\theta} t^2 ; t \in \mathbb{R} \}$
- $\sigma(A_\alpha) = \sigma_{ess}(A_\alpha) \cup \sigma_{disc}(A_\alpha)$
- $\sigma(A_\alpha) \subset \{ z \in \mathbb{C} ; -2\theta < \text{arg}(z) \leq 0 \}$

(see Kalvin, Kim and Pasciak)

---

![Graph](https://via.placeholder.com/150)
Some elements of proof

Proof of the second item:

\[ \sigma_{\text{ess}}(A_{\alpha}) = \sigma_{\text{ess}}(-\Delta_{\theta}) \]

\[ = \bigcup_{n \geq 0} \sigma_{\text{ess}}(-\Delta^{(n)}_{\theta}) \]

\[ = \bigcup_{n \geq 0} \left\{ n^2 \pi^2 + e^{-2i\theta} t^2; t \in \mathbb{R} \right\} \]

Essential spectrum of \( A_{\alpha} \):

\[ \Delta_{\theta} = e^{-2i\theta} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

\[ \Delta^{(n)}_{\theta} = e^{-2i\theta} \frac{\partial^2}{\partial x^2} + n^2 \pi^2 \]
Some elements of proof

**Proof of the third item:** \( \sigma(A_\alpha) = \sigma_{ess}(A_\alpha) \cup \sigma_{disc}(A_\alpha) \)

The result follows from analytic Fredholm theorem because:

1. \( U = \mathbb{C} \setminus \sigma_{ess}(A_\alpha) \) is a connected set.
2. There is a point \( z \in U \) such that \( A_\alpha - z \) is invertible (coerciveness).

(See D.E. Edmunds and W.D. Evans, Spectral theory and differential operators.)
Trapped modes correspond to **discrete** real eigenvalues of $A_\alpha$!

Other eigenvalues correspond to **complex resonances**, with a field $u$ exponentially growing at infinity.

**Spectrum of $A_\alpha$:**

![Diagram](image-url)
**Numerical Illustration**

The numerical results have been obtained by a finite element discretization with FreeFem++.  

Here the scatterer is a non-penetrable rectangular obstacle in the middle of the waveguide:

We put PMLs in the magenta parts:
The numerical results have been obtained by a finite element discretization with **FreeFem++**.

Here the scatterer is a **non-penetrable rectangular obstacle** in the middle of the waveguide:

We put **PMLs** in the magenta parts:

In the next slides, we represent the **square-root of the spectrum**, which corresponds to \( k \) values.
Numerical illustration
Numerical illustration
There are two trapped modes:
1. A main tool: the complex scaling (PML)

2. Spectrum of trapped modes frequencies

3. Spectrum of no-reflection frequencies

4. Extensions and comments
A new complex spectrum linked to $\mathcal{H}$ with ”conjugate” PMLs

A simple and important remark

For $k \in \mathcal{H}$, the total field is ingoing at the left-hand side of $\mathcal{O}$ and outgoing at the right-hand side of $\mathcal{O}$.

The idea is to use a complex scaling (and numerically PMLs), with complex conjugate parameters at both sides of the obstacle, so that the transformed total field $u$ will belong to $L^2(\Omega)$. 
A NEW COMPLEX SPECTRUM LINKED TO $\mathcal{H}$
WITH "CONJUGATE" PMLs

Let us consider now the following unbounded operator:

$$D(A\tilde{\alpha}) = \{ u \in L^2(\Omega); A\tilde{\alpha}u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \}$$

$$A\tilde{\alpha}u = -\frac{1}{1 + \rho(x, y)} \left( \tilde{\alpha}(x) \frac{\partial}{\partial x} \left( \tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)$$

\[ \tilde{\alpha}(x) = e^{i\theta} \quad \tilde{\alpha}(x) = 1 \quad \tilde{\alpha}(x) = e^{-i\theta} \]
A new complex spectrum linked to $\mathcal{H}$
with "conjugate" PMLs

Let us consider now the following unbounded operator:

$$D(A_{\tilde{\alpha}}) = \{ u \in L^2(\Omega); A_{\tilde{\alpha}} u \in L^2(\Omega); \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \}$$

$$A_{\tilde{\alpha}} u = -\frac{1}{1 + \rho(x, y)} \left( \tilde{\alpha}(x) \frac{\partial}{\partial x} \left( \tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)$$

**Spectral features of $A_{\tilde{\alpha}}$**

- $A_{\tilde{\alpha}}$ is a non self-adjoint operator.
- $\sigma_{ess}(A_{\tilde{\alpha}}) = \bigcup_{n \geq 0} \{ n^2 \pi^2 + e^{2i\theta} t^2; t \in \mathbb{R} \} \cup \{ n^2 \pi^2 + e^{-2i\theta} t^2; t \in \mathbb{R} \}$
- $\sigma_{disc}(A_{\tilde{\alpha}}) \subset \{ z \in \mathbb{C}; -2\theta < \arg(z) < 2\theta \}$
A NEW COMPLEX SPECTRUM LINKED TO $\mathcal{H}$
with "conjugate" PMLs

Typical expected spectrum of $A_{\tilde{\alpha}}$:

\[ \sigma_{\text{ess}}(A_{\tilde{\alpha}}) = \bigcup_{n \geq 0} \{n^2 \pi^2 + e^{2i\theta} t^2; t \in \mathbb{R}\} \cup \{n^2 \pi^2 + e^{-2i\theta} t^2; t \in \mathbb{R}\} \]
\[ \sigma(A_{\tilde{\alpha}}) \subset \{z \in \mathbb{C}; -2\theta < \text{arg}(z) < 2\theta\} \]
A new complex spectrum linked to $\mathcal{K}$ with "conjugate" PMLs

Typical expected spectrum of $A_{\tilde{\alpha}}$:

Difficulty: $\mathbb{C} \setminus \sigma_{\text{ess}}(A_{\tilde{\alpha}})$ is not a connected set.

Conjecture

$$\sigma(A_{\tilde{\alpha}}) = \sigma_{\text{ess}}(A_{\tilde{\alpha}}) \cup \sigma_{\text{disc}}(A_{\tilde{\alpha}}) \text{ if } \rho \neq 0$$
Pathological cases

In the unperturbed case ($\rho = 0$):

All $k^2$ in the yellow zone are eigenvalues of $A_{\tilde{\alpha}}$!

Proof: Use the stretched plane wave as an eigenvector:

$$A_{\tilde{\alpha}} u = k^2 u$$

for $u(x, y) = \begin{cases} 
  e^{ik(-R+(x+R)e^{-i\theta})} & \text{if } x < -R \\
  e^{ikx} & \text{if } -R < x < R \\
  e^{ik(R+(x-R)e^{i\theta})} & \text{if } R < x 
\end{cases}$
Pathological cases

And the same result holds with horizontal cracks!

All $k^2$ in the yellow zone are eigenvalues of $A_{\tilde{\alpha}}$!

**Proof:** Use the stretched plane wave as an eigenvector:

$$A_{\tilde{\alpha}}u = k^2 u$$

for $u(x, y) = \begin{cases} 
  e^{ik(-R+(x+R)e^{-i\theta})} & \text{if } x < -R \\
  e^{ikx} & \text{if } -R < x < R \\
  e^{ik(R+(x-R)e^{i\theta})} & \text{if } R < x 
\end{cases}$
For real eigenvalues, the eigenmode is such that

- $u$ is ingoing
- $\mathcal{O}$ $u$ is outgoing
For $k^2 \in \sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R}$, the eigenmode is such that:

- $u$ is ingoing
- $u$ is outgoing

There are two cases:

- Either $u$ on the left-hand side contains a propagating part and it is a case of no-reflection: $k \in \mathcal{H}$.
- Either $u$ is evanescent on both sides and $k$ is associated to a trapped mode: $k \in \mathcal{T}$.

**Theorem**

$$\sigma_{disc}(A_{\tilde{\alpha}}) \cap \mathbb{R} = \{ k^2 \in \mathbb{R}; k \in \mathcal{H} \cup \mathcal{T} \}$$
Remember that:

\[
A \tilde{\alpha} u = -\frac{1}{1 + \rho(x, y)} \left( \tilde{\alpha}(x) \frac{\partial}{\partial x} \left( \tilde{\alpha}(x) \frac{\partial u}{\partial x} \right) + \frac{\partial^2 u}{\partial y^2} \right)
\]

and that

\[
\tilde{\alpha}(-x) = \overline{\tilde{\alpha}(x)}
\]

**Consequence**

If the obstacle is symmetric in \(x\):

\[
\rho(-x, y) = \rho(x, y)
\]

\(A \tilde{\alpha}\) is \(\mathcal{PT}\)-symmetric and its spectrum is stable by complex conjugation:

\[
\sigma(A \tilde{\alpha}) = \overline{\sigma(A \tilde{\alpha})}
\]
The spectrum is symmetric w.r.t. the real axis ($\mathcal{PT}$-symmetry).
There are much more real eigenvalues than for trapped modes.
NUMERICAL ILLUSTRATION
FOR A RECTANGULAR SYMMETRIC CAVITY

Red: classical PMLs
Blue: conjugate PMLs
Numerical illustration for a rectangular symmetric cavity

This is a representation of the computed modes for the 10 first real eigenvalues and in the whole computational domain (including PMLs).
Let us focus on the eigenmodes such that $0 < k < \pi$:

First trapped mode: 
$k = 1.2355 \cdots$

Second trapped mode: 
$k = 2.3897 \cdots$

First no-reflection mode: 
$k = 1.4513 \cdots$

Second no-reflection mode: 
$k = 2.8896 \cdots$
To validate this result, we compute the amplitude of the reflected plane wave for $0 < k < \pi$:

First no-reflection mode: 
$k = 1.4513 \cdots$

Second no-reflection mode: 
$k = 2.8896 \cdots$

There is a **perfect agreement**!
No-reflection mode in the time-domain

Below we represent $\Re(e^{i\omega t})$ with $u$...

...a no-reflection mode:

![Image of no-reflection mode]

with the corresponding incident propagating mode:

![Image of incident propagating mode]

We observe no reflection but a phase shift in the transmitted wave.
NO-REFLECTION MODE IN THE TIME-DOMAIN

Below we represent $\Re(e^{i\omega t})$ with $u$...

...a no-reflection mode:

with the corresponding incident propagating mode:

We observe no reflection but a phase shift in the transmitted wave.
Numerical illustration
in a non $\mathcal{PT}$-symmetric case

Here the scatterer is a not symmetric in $x$, and neither in $y$:

We expect:
- No trapped modes
- No invariance of the spectrum by complex conjugation
The spectrum is no longer symmetric w.r.t. the real axis.

There are several eigenvalues near the real axis.
Numerical illustration in a non $\mathcal{PT}$-symmetric case

Again results can be validated by computing $R(k)$ for $0 < k < \pi$:

\[
\begin{align*}
  k &= 1.2803 + 0.0003i &
  k &= 2.3868 + 0.0004i &
  k &= 2.8650 + 0.0241i \\
\end{align*}
\]

Complex eigenvalues also contain useful information about almost no-reflection.
Outline

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Multiport waveguides junction

**OBJECTIVE**

Find \((k, u)\) such that \(u\) is ingoing in some ports and outgoing in the others.

For an \(N\)-ports junction, there are \(2^{N-1}\) such problems and corresponding spectra.
This is a bar-bar example of such problem:

There are two axes of $\mathcal{PT}$-symmetry!
An interesting configuration is the junction of 2 different Dirichlet waveguides.

**Consequences**

- Now $\mathbb{C}\setminus \sigma_{ess}(A_\tilde{\alpha})$ is a connected set!
- Our "new" eigenvalues correspond in fact to classical complex resonances in non-classical sheets of the Riemannn surface......
**Conclusion**

There is still a lot of work to do!

- **Clarify the link** between our new spectrum and classical resonance frequencies.
- **Prove the existence of no-reflection frequencies** ($\mathcal{H} \neq \emptyset$), at least in $\mathcal{PT}$-symmetric cases.
- **Justify the numerics** (absence of spectral pollution).
- **Find similar spectral approaches** for other phenomena in waveguides (perfect invisibility, total reflection, modal conversion, etc...)

...  

These results have been published in:
