Sharp time decay estimates for dispersive equations

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Joint work with Nicolas Burq

2 Octobre 2018,Toulouse

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These time decay rates are **sharp**.

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Follows from the Huygens Principle, i.e.

$$\mathrm{supp}\Big(\big[\cos(t|D|)\big]\Big)\subset\Big\{(x,y)\ :\ |x-y|=|t|\Big\}.$$

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Decay in t of $e^{-itP} \longleftrightarrow$ **Smoothness** in λ of the spectral measure dE_λ.

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  - Bony-Häfner (2012): ϵ sharp estimates
  - Guillarmou-Hassell-Sikora (2013): sharp estimates for *scattering metrics* (polyhomogeneous expansion of the metric)

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Analogous for n even.

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with $c > 0$ and $\phi \in C_0^\infty(\mathbb{R})$, $\phi \equiv 1$ near 1.

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Lemma. If $s \in [0, \frac{n}{4}]$, $\sigma > 2s$ and $\kappa > s$,

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$$||u||_{L^2}^{1+\frac{2}{n}} \lesssim ||u||_{L^1}^{\frac{2}{n}} ||\nabla u||_{L^2} \lesssim ||u||_{L^1}^{\frac{2}{n}} ||\sqrt{P}u||_{L^2}$$

it is well known that one gets

$$||e^{-tP}||_{L^p \to L^2} \lesssim t^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}.$$

Lemma. If $s \in [0, \frac{n}{4}]$, $\sigma > 2s$ and $\kappa > s$,

$$\left|\left|(P/\lambda + 1)^{-\kappa} \langle x \rangle^{-\sigma}\right|\right|_{L^2 \to L^2} \lesssim \lambda^s.$$

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Proof:

$$(P/\lambda + 1)^{-\kappa} = \frac{1}{\Gamma(\kappa)} \int_0^{+\infty} e^{-t(P/\lambda+1)} t^{\kappa-1} dt. \ \square$$

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Rem. This also gives *elliptic* low frequency resolvent estimates

$$\left|\left|(P + \lambda)^{-\kappa} \langle x \rangle^{-\sigma}\right|\right|_{L^2 \to L^2} \lesssim \lambda^{s-\kappa}.$$

First application

Assume we proved that for $\nu > \max\{k, \frac{n}{2}\}$ and $F \in C_0^\infty(\mathbb{R})$, $F \equiv 1$ on $[0, 10]$,

$$\big|\big|\langle x\rangle^{-\nu} F(P/\lambda)\big(P - \lambda \mp i0\big)^{-k} \langle x\rangle^{-\nu}\big|\big|_{\mathcal{L}(L^2)} \lesssim \lambda^{\frac{n}{2}-k}$$

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If $\tilde{G} \equiv 1$ near the support of G, one can write

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Positive commutators and resolvent (E. Mourre - proof by C. Gérard)

Assume that for some self-adjoint operators A and H,

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$$||\langle \varepsilon A\rangle^{-s} u|| \ ||\langle \varepsilon A\rangle^{s}(H-z)u|| \geq \mathrm{Im}\Big(\Theta(\varepsilon A)u, (H-z)u\Big) \geq c\varepsilon ||\langle \varepsilon A\rangle^{-s}u||^2 - C\varepsilon^2 ||\langle \varepsilon A\rangle^{-s} u||^2$$

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$$||\langle\varepsilon A\rangle^{-s}u||\;||\langle\varepsilon A\rangle^{s}(H-z)u|| \geq \mathrm{Im}\Big(\Theta(\varepsilon A)u, (H-z)u\Big) \geq c\varepsilon||\langle\varepsilon A\rangle^{-s}u||^2 - C\varepsilon^2||\langle\varepsilon A\rangle^{-s}u||^2$$

i.e. for some constant C_ε **independent of δ**

$$||\langle\varepsilon A\rangle^{-s}u|| \leq C_\varepsilon||\langle\varepsilon A\rangle^{s}(H-z)u|| \;\;\Rightarrow\;\; ||\langle\varepsilon A\rangle^{-s}\psi(H)(H-z)^{-1}\langle\varepsilon A\rangle^{-s}|| \leq C'_\varepsilon$$

Some insights on the proof of Theorem 1

We wish to apply the positive commutator method to

$$\frac{P}{\lambda} = S_\lambda P_\lambda S_\lambda^{-1}, \qquad P_\lambda = g^{jk}(\lambda^{-\frac{1}{2}}y)\partial_j\partial_k + \lambda^{-\frac{1}{2}} b_j(\lambda^{-\frac{1}{2}}y)\partial_j$$

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The operator P_λ has **singular coefficients at** 0. In fact, if $|y| \gtrsim 1$,

$$\begin{aligned} \left| \partial_y^\alpha \left(g^{jk}(\lambda^{-\frac{1}{2}} y) - \delta_{jk} \right) \right| \quad & \lesssim \quad \lambda^{-\frac{|\alpha|}{2}} \langle \lambda^{-\frac{1}{2}} y \rangle^{-\rho - |\alpha|} \\ & \lesssim \quad \lambda^{-\frac{|\alpha|}{2}} \left| \lambda^{-\frac{1}{2}} y \right|^{-\rho - |\alpha|} = \lambda^{\frac{\rho}{2}} |y|^{-\rho - |\alpha|} \lesssim \lambda^{\frac{\rho}{2}} \langle y \rangle^{-\rho - |\alpha|} \end{aligned}$$

(similar estimates for $\lambda^{-\frac{1}{2}} b_j(\lambda^{-\frac{1}{2}} y)$)

Proposition 1 If $\zeta \in C^\infty(\mathbb{R}^n)$ vanishes near 0,

$$\zeta(\lambda^{\frac{1}{2}} x)\big(P/\lambda + 1\big)^{-1} \sim S_\lambda q_\lambda(x, D) S_\lambda^{-1}$$

for some bounded family $(b_\lambda)_{\lambda \ll 1}$ of S^{-2}.

Some insights on the proof of Theorem 1

Assume for simplicity that $|g(x)| \equiv 1$. If $\chi \in C_0^\infty(\mathbb{R}^n \setminus 0)$ eqals 1 near infinity, we set

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Proposition 2 As $\lambda \to 0$ and $\mathrm{supp}(\varphi) \to \{1\}$,

$$\Big|\Big|\varphi(P/\lambda)\langle \lambda^{\frac{1}{2}} x\rangle^{-\rho}\Big|\Big| \to 0.$$

By Proposition 2, for $\lambda \ll 1$ and φ supported close enough to 1,

$$\varphi(P/\lambda) i\big[(P/\lambda), A] \varphi(P/\lambda) \quad \geq \quad \frac{3}{2}\varphi^2(P/\lambda) - C\Big|\Big|\varphi(P/\lambda)\langle \lambda^{\frac{1}{2}} x\rangle^{-\rho}\Big|\Big|$$

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Selecting ϕ such that $\phi\varphi = \phi$ and $\phi \equiv 1$ near 1,

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The proof of Proposition 2 uses the next lemma lemma, based on heat flow estimates.
Proposition 3 There exists $\delta > 0$ such that for all $\varphi \in C_0^\infty(\mathbb{R})$

$$\big|\big|\varphi(P/\lambda) - \varphi(-\Delta/\lambda)\big|\big|_{L^2 \to L^2} \lesssim_\varphi \lambda^\delta$$