Orbital stability of standing wave solutions of a Klein-Gordon equation with Dirac delta potentials

Elek Csobo

TU Delft

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joint work with F. Genoud, M. Ohta and J. Royer

Introduction

We examine the well-posedness and orbital stability of standing wave solutions of the following nonlinear Klein-Gordon equation (NKG) with Dirac delta potentials:

$$\begin{cases} u_{tt} - u_{xx} + m^2 u + \gamma\delta u + \mathrm{i}\alpha\delta u_t - |u|^{p-1}u = 0, \\ u(t,x) \to 0, \quad \text{as} \quad |x| \to \infty, \\ (u(t), \partial_t u(t))|_{t=0} = (u_0, u_1), \end{cases}$$

where δ is the Dirac mass at $x = 0$, $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}$, $m > 0$ and $p > 1$. We seek complex-valued solutions to the initial-value problem.

Introduction

- It is convenient to reformulate the initial-value problem as a first order system for the variables $(u, v) = (u, u_t)$.
- We seek solutions in $\mathscr{H} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, which we regard as a real Hilbert space, endowed with the inner product

$$\langle (u_1, v_1), (u_2, v_2) \rangle_{\mathscr{H}} = \langle u_1', u_2' \rangle_{L^2} + \langle u_1, u_2 \rangle_{L^2} + \langle v_1, v_2 \rangle_{L^2},$$

where the real L^2 inner product is defined as

$$\langle u, v \rangle_{L^2} = \mathrm{Re} \int_{\mathbb{R}} u \bar{v} \, \mathrm{d}x.$$

Introduction

We interpret $-\partial_x^2 + \beta\delta$ on $L^2(\mathbb{R})$ as the operator H_β, defined by

$$
\begin{aligned}
&-\partial_x^2 : D_\beta \to L^2(\mathbb{R}),\\
&\textit{where} \quad D_\beta = \{u \in H^2(\mathbb{R}\setminus\{0\}) \cap H^1(\mathbb{R}) \,:\, u'(0^+) - u'(0^-) = \beta u(0)\}.
\end{aligned}
$$

This is justified with the following short calculations: for $u, v \in H^1(\mathbb{R})$

$$
\langle(-\partial_x^2 + \beta\delta)u, v\rangle_{H^{-1},H^1} = \mathrm{Re}\int_{\mathbb{R}} \partial_x u \partial_x \bar{v}\,\mathrm{d}x + \beta\,\mathrm{Re}\,u(0)\bar{v}(0)
$$

on the other hand for $u \in D_\beta$ and $v \in H^1$

$$
\begin{aligned}
\langle H_\beta u, v\rangle &= \mathrm{Re}\int_{\mathbb{R}} -\partial_x^2 u\bar{v}\,\mathrm{d}x = \mathrm{Re}\left(\int_{-\infty}^{0} -\partial_x^2 u\bar{v} + \int_0^{+\infty} -\partial_x^2 u\bar{v}\right)\\
&= \mathrm{Re}\int_{\mathbb{R}} \partial_x u\partial_x\bar{v}\,\mathrm{d}x + \mathrm{Re}((u'(0^+) - u'(0^-))\bar{v}(0))\\
&= \mathrm{Re}\int_{\mathbb{R}} \partial_x u\partial_x\bar{v}\,\mathrm{d}x + \beta\,\mathrm{Re}\,u(0)\bar{v}(0).
\end{aligned}
$$

Well-posedness of the Cauchy problem

Being a solution of (NKG) with Dirac potentials amounts to being a solution of the unperturbed equation, and satisfying the jump condition

$$u'(0^+) - u'(0^-) = \gamma u(0) + \mathrm{i}\alpha v(0).$$

The influence of the Dirac potentials will be encoded in the domain of the operator

$$\mathcal{A} = \begin{pmatrix} 0 & Id \\ \partial_x^2 - m^2 & 0 \end{pmatrix},$$

which is acting on

$$\begin{aligned} \mathfrak{D} = \{(u,v) \in \big(H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{0\})\big) \times H^1(\mathbb{R}) :& \\ u'(0^+) - u'(0^-) = \gamma u(0) + \mathrm{i}\alpha v(0)\} \subset H^1 \times L^2.& \end{aligned}$$

Well-posedness of the Cauchy problem

- Introducing $U = (u, v)$ and $F(U) = (0, |u|^{p-1}u)$, the first order system corresponding to (NKG) reads

$$\begin{cases} U_t(t) - \mathcal{A}U(t) = F(U), \\ U(0) = U_0. \end{cases}$$

- The operator $(\mathcal{A}, \mathfrak{D})$ generates a C^0-semigroup on $\mathscr{H}$ and we can prove local well-posedness for (NKG) by applying standard results of semigroups to the problem

$$U(t) = \mathrm{e}^{t\mathcal{A}}U_0 - \int_0^t \mathrm{e}^{(t-s)\mathcal{A}}F(U(s))\,\mathrm{d}s.$$

Well-posedness of the Cauchy problem

Theorem (Local well-posedness of the Cauchy problem)

Let $U_0 \in \mathscr{H}$. Then there exists $T(U_0) \in (0, \infty]$ such that the following statements are true.

- *There exists $U \in C^0([0, T(U_0)), \mathscr{H})$ such that U is a maximal unique solution of (NKG).*
- *If $T(U_0) < \infty$, then $\|U(t)\|_{\mathscr{H}} \to +\infty$ as $t \to T(U_0)$.*

Well-posedness of the Cauchy problem

Additionally, for any initial data $(u, v) \in \mathscr{H}$, the **energy** and the **charge** are constant along the flow of the equation

$$E(u,v) = \frac{1}{2}\|u'\|_{L^2}^2 + \frac{m^2}{2}\|u\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + \frac{\gamma}{2}|u(0)|^2 - \frac{1}{p+1}\int_{\mathbb{R}} |u|^{p+1}\,\mathrm{d}x,$$

$$Q(u,v) = \mathrm{Im}\int_{\mathbb{R}} u\bar{v}\,\mathrm{d}x - \frac{\alpha}{2}|u(0)|^2.$$

Hamiltonian structure of the system

It is possible to show that (NKG) is a Hamiltonian system. Here we follow the framework of the paper by De Bièvre, Genoud and Rota Nodari. Let us define the map $\mathcal{J} : \mathscr{H} \to \mathscr{H}^*$ as

$$\mathcal{J}(u, v) = (-\mathrm{i}\alpha u\delta - v, u).$$

We have established the following lemma:

Lemma

The map $\mathcal{J} : \mathscr{H} \to \mathscr{H}^*$ is a symplector, that is:

- *$\mathcal{J}$ is a bounded linear map;*
- *$\mathcal{J}$ is one-to-one;*
- *$\mathcal{J}$ is anti-symmetric, in the sense that*

$$\langle \mathcal{J}(u, v), (\varphi, \psi)\rangle = -\langle \mathcal{J}(\varphi, \psi), (u, v)\rangle, \quad (u, v), (\varphi, \psi) \in \mathscr{H}.$$

Hamiltonian structure of the system

- From routine calculations it follows that $E \in C^2(\mathscr{H}, \mathbb{R})$ and $Q \in C^2(\mathscr{H}, \mathbb{R})$.
- Indeed, $(\mathscr{H}, \mathfrak{D}, \mathcal{J})$ is a suitable symplectic Banach triple for our problem, with associated Hamiltonian E, and the differential equation can be written as the Hamiltonian system

$$\mathcal{J} \frac{\mathrm{d}}{\mathrm{d}t} U(t) = E'(U(t)).$$

Orbital stability of standing waves

We consider standing wave solutions of the initial-value problem (NKG) of the form

$$u(t,x) = \mathrm{e}^{\mathrm{i}\omega t}\varphi(x).$$

This Ansatz leads to the standing wave equation

$$-\varphi'' + (m^2 - \omega^2)\varphi + (\gamma - \alpha\omega)\delta\varphi - |\varphi|^{p-1}\varphi = 0.$$

Introducing $U_\omega(t,x) = \mathrm{e}^{\mathrm{i}\omega t}(\varphi_\omega(x), \mathrm{i}\omega\varphi_\omega(x))$, the standing wave equation in the Hamiltonian setting becomes:

$$E'(U_\omega) + \omega Q'(U_\omega) = 0.$$

Hence standing wave solutions are exactly the critical points of the **momentum map**

$$L(u,v) = E(u,v) + \omega Q(u,v).$$

Orbital stability of standing waves

Definition

A standing wave is **orbitally stable** if for any $\epsilon > 0$ there is a $\delta > 0$, such that if $U(t)$ is a solution of (NKG), then we have that

$$\|U(0) - (\varphi_\omega, \mathrm{i}\omega\varphi_\omega)\|_{\mathscr{H}} < \delta \Rightarrow \sup_{\theta\in\mathbb{R}} \|U(t) - \mathrm{e}^{i\theta}(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)\|_{\mathscr{H}} < \epsilon$$

for all $t \in \mathbb{R}$.

Orbital stability of standing waves

The general approach to study orbital stability was developed in the seminal papers by Grillakis, Shatah and Strauss, newly revisited by De Bièvre et al. Roughly speaking, to determine orbital stability of the standing waves (under some general assumptions) it suffices to examine the spectral properties of the linear operator

$$L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega) : \mathscr{H} \to \mathscr{H}^*,$$

and the sign of the derivative

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\Big|_{\omega=\omega_0} Q(U_\omega),$$

where Q is the charge

$$Q(U_\omega) = -\omega\|\varphi_\omega\|_{L^2}^2 - \frac{\alpha}{2}|\varphi_\omega(0)|^2.$$

Orbital stability of standing waves

Proposition (Grillakis et al. (1987))

If $n(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)) = 1$, 0 *is a simple eigenvalue of* $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ *and the rest of the spectrum is positive and bounded away from* 0, *then the standing wave* U_ω *is orbitally stable if*

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\bigg|_{\omega=\omega_0} Q(U_\omega) > 0,$$

and orbitally unstable if

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\bigg|_{\omega=\omega_0} Q(U_\omega) < 0.$$

Notation: $n(L)$ denotes the number of negative eigenvalues of the operator L.

Orbital stability of standing waves

Proposition (Grillakis et al. 1990)

- *If* $n(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)) = 2$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\bigg|_{\omega=\omega_0} Q(U_\omega) > 0,$$

  then the standing wave U_ω *is orbitally unstable.*

- *If* $n(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)) = 2$ *and* $\frac{\mathrm{d}}{\mathrm{d}\omega}\Big|_{\omega=\omega_0} Q(U_\omega) < 0$, *then the standing wave* U_ω *is orbitally unstable in* $\mathscr{H}_{\mathrm{rad}}(\mathbb{R})$, *which implies instability in* $\mathscr{H}(\mathbb{R})$.

Spectral properties of the linearized operator

- Since the solution of the standing wave equation φ_ω depends on the parameters α, γ, and ω, the linearized operator $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ also depends on them. To examine the dependence of stability with respect to the parameters α and γ, we shall pursue a perturbative method for a fixed value of ω.
- To carry out the spectral analysis of $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ it is convenient to split u and v into real and imaginary parts. We rewrite $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ as an operator acting on

$$L^2(\mathbb{R},\mathbb{R}) \times L^2(\mathbb{R},\mathbb{R}) \times L^2(\mathbb{R},\mathbb{R}) \times L^2(\mathbb{R},\mathbb{R}).$$

Spectral conditions

Hence we can write for $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ the following:

$$\begin{bmatrix} L_1 & 0 & 0 & -\omega \\ 0 & L_2 & \omega & 0 \\ 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \end{bmatrix},$$

where

$$L_1 = -\partial_x^2 + m^2 - p\varphi_\omega^{p-1} + (\gamma - \alpha\omega)\delta,$$
$$L_2 = -\partial_x^2 + m^2 - \varphi_\omega^{p-1} + (\gamma - \alpha\omega)\delta.$$

To solve the $(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega) - \lambda)(u, v) = (f, g)$ we can eliminate the terms v_1 and v_2, which are the real and imaginary parts of v.

Spectral conditions

Having done the necessary eliminations we obtain

$$L^+_{\alpha,\gamma} u_1 + \mathrm{i} L^-_{\alpha,\gamma} u_2,$$

where $u = u_1 + \mathrm{i} u_2$. $L^+_{\alpha,\gamma}$ and $L^-_{\alpha,\gamma}$ are the well-known Schrödinger operators:

$$\begin{aligned} L^+_{\alpha,\gamma} &= -\partial_x^2 + (m^2 - \omega^2) - p\varphi^{p-1}_{\alpha,\gamma} + (\alpha - \gamma\omega)\delta, \\ L^-_{\alpha,\gamma} &= -\partial_x^2 + (m^2 - \omega^2) - \varphi^{p-1}_{\alpha,\gamma} + (\alpha - \gamma\omega)\delta. \end{aligned}$$

Spectral conditions

Lemma

Let $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}$ *and* $4(m^2 - \omega^2) > (\gamma - \alpha\omega)^2$. *Then*

- *The operator* $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ *has finite number of negative eigenvalues.*
- *The kernel of* $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ *is spanned by* $\mathrm{i}\varphi_{\alpha,\gamma}$.
- *The essential spectrum of* $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ *positive and bounded away from* 0.

Additionally from the proof we obtain that $n(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)) = n(L^+_{\alpha,\gamma})$, since the first eigenvalue of $L^-_{\alpha,\gamma}$ is 0, and the corresponding eigenvector is $\varphi_{\alpha,\gamma}$.

Introducing $\beta = -(\gamma - \alpha\omega)$, we get $(L^+_\beta)_{\beta\in\mathbb{R}}$ a holomorphic family of self-adjoint operators. The $\beta = 0$ case is covered by a well-known lemma by Weinstein (1985).

Spectral conditions

Carrying out the perturbative argument we obtain the following proposition:

Proposition

Let $4(m^2 - \omega^2) > (\gamma - \alpha\omega)^2$. *The following statements hold:*

- *If* $(\gamma - \alpha\omega) > 0$ *then* $n(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)) = 2$.
- *If* $(\gamma - \alpha\omega) < 0$ *then* $n(L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)) = 1$.

Remark: If $(\gamma - \alpha\omega) > 0$ and $L''(\varphi_\omega, \mathrm{i}\omega\varphi_\omega)$ is restricted to radial functions it has one negative eigenvalue. Instability to radial perturbations implies instability to general perturbations.

Slope condition

Having the explicit formula for the standing wave we can make explicit calculations to investigate slope condition. Specifically, for $p = 3$ we have the following algebraic expression for the derivative of $Q(U_\omega)$:

$$\frac{\mathrm{d}}{\mathrm{d}\omega} Q(U_\omega) = \frac{4\omega^2}{\sqrt{m^2 - \omega^2}} - 4\sqrt{m^2 - \omega^2} + \left(\frac{\alpha^3}{2} + 6\alpha\right)\omega - \gamma\left(2 + \frac{\alpha^2}{2}\right).$$

For fixed ω and α, we have $\tilde{\gamma}(\alpha, \omega)$ (explicitly known), such that for $\gamma \in \mathbb{R}$ we have

$$\mathrm{sgn}\frac{\mathrm{d}}{\mathrm{d}\omega} Q(U_\omega) = \mathrm{sgn}(\gamma - \tilde{\gamma}).$$

Stability theorem

From the spectral and slope conditions we obtain the following theorem:

Theorem

Let $p = 3$, $\omega \in (-m, m)$, and $\gamma \in \mathbb{R}$, such that $4(m^2 - \omega^2) > (\gamma - \alpha\omega)^2$ is satisfied. There exists $\tilde{\gamma}(\alpha, \omega)$ function explicitly known such that the following assertions are true:

- *If $\tilde{\gamma} < \gamma$ and $(\gamma - \alpha\omega) < 0$, the standing wave $e^{i\omega t}\varphi(x)$ is orbitally stable.*
- *If $\tilde{\gamma} > \gamma$, the standing wave $e^{i\omega t}\varphi(x)$ is unstable.*
- *If $\tilde{\gamma} < \gamma$ and $(\gamma - \alpha\omega) > 0$, the standing wave $e^{i\omega t}\varphi(x)$ is unstable.*

Thanks for your attention!