Exterior energy bounds for wave equations and applications

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Outline

1. Energy-critical wave equation

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2. Radiation terms and exterior energy bounds

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3. Nonradiative solutions and rigidity theorems for critical wave equations

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4. Outgoing initial data

The equation

(NLW) $$\begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u \\ \vec{u}_{\restriction t=0} = (u_0, u_1) \in \mathcal{H} = \dot{H}^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N) \end{cases}$$

where $u : [0, T[\times \mathbb{R}^N \to \mathbb{R}, \quad N \geq 3.$

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The energy

$$E(\vec{u}) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_{t,x} u(t)|^2 - \frac{N-2}{2N} \int_{\mathbb{R}^N} |u(t)|^{\frac{2N}{N-2}}$$

is conserved, where $|\nabla_{t,x} u|^2 = (\partial_t u)^2 + \sum_{j=1}^N (\partial_{x_j} u)^2$.

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Transformations: if u is a solution and $\lambda > 0, \; t_0 \in \mathbb{R}, \; \varepsilon \in \{\pm 1\}$, so is

$$v(t,x) = \frac{\varepsilon}{\lambda^{N/2-1}} u\left(\frac{t-t_0}{\lambda}, \frac{x-x_0}{\lambda}\right).$$

$$\|\vec{v}(t_0)\|_{\mathcal{H}} = \|\vec{u}(0)\|_{\mathcal{H}}, \quad E(\vec{v}) = E(\vec{u}).$$

Travelling waves

Stationary solutions of (NLW):

(E) $$-\Delta Q = |Q|^{\frac{4}{N-2}} Q, \quad Q : \mathbb{R}^N \to \mathbb{R}, \quad Q \in \dot{H}^1(\mathbb{R}^N).$$

Minimal energy nonzero solution of (E) (ground state):

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{1-\frac{N}{2}}$$

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Solitary waves ($\mathbf{p} \in \mathbb{R}^N$, $|\mathbf{p}| < 1$): $Q_{\mathbf{p}}(t, x) = Q_{\mathbf{p}}(0, x - t\mathbf{p})$, where

$$Q_{\mathbf{p}}(0, x) = Q\left(\left(\frac{1}{\sqrt{1-p^2}} - 1\right)\frac{(\mathbf{p}\cdot x)\mathbf{p}}{p^2} + x\right)$$

Type II blow-up solutions

Definition. *u is a type II blow-up solution of* (NLW) *when the maximal time of existence* T_+ *of u is finite and*

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Existence of solutions of the form:

$$\vec{u}(t) = \left(\frac{1}{\lambda(t)^{\frac{N-2}{2}}} W\left(\frac{\cdot}{\lambda(t)}\right), 0\right) + (v_0, v_1), \quad t \to T_+,$$

where $(v_0, v_1) \in \mathcal{H}$ et $\lambda(t) \ll T_+ - t$, [Krieger Schlag Tataru 09, 14] ($N = 3$) and also: [Hillairet Raphaël 12] ($N = 4$), [Jendrej 2015].

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Open question: Type II solutions with more than one bubble, or other bubbles than W. See: [Jendrej], [Martel, Merle] (global case) and also [Côte, Zaag] (1D), [Côte, Martel] (KG).

Soliton resolution conjecture for type II blow-up

Conjecture. *Let u be a radial, type II blow-up solution of* (NLW)*. Then there exists $J \geq 1$ and:*

- $(v_0, v_1) \in \mathcal{H}$,
- *signs* $\varepsilon_j \in \{\pm 1\}$, $j = 1 \dots J$,
- *parameters* $\lambda_j(t)$, $0 < \lambda_1(t) \ll \lambda_2(t) \ll \dots \ll \lambda_J(t) \ll T_+ - t$,

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In the nonradial case: possibly several blow-up points, more general solitons (replace $\pm W$ by $Q^j_{\mathbf{p}_j}$), space translations allowed.

Outline

Asymptotic behaviour for linear waves

Consider:

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Then (see e.g. [Friedlander]) there exist $G_\pm \in L^2(\mathbb{R} \times S^{N-1})$ such that:

$$\lim_{t\to+\infty} \int_0^{+\infty} \int_{S^{N-1}} \left| r^{\frac{N-1}{2}} \partial_r u_L(t, r\omega) \mp G_\pm(r-t, \omega) \right|^2$$

$$+ \left| r^{\frac{N-1}{2}} \partial_t u_L(t, r\omega) + G_\pm(r-t, \omega) \right|^2 dr d\omega = 0.$$

Furthermore (denoting $\not\partial$ the tangential derivative):

$$\lim_{t\to+\infty} \int \frac{1}{|x|^2} |u_L(t,x)|^2 + |\not\partial u_L(t,x)|^2 + |u_L(t,x)|^{\frac{2N}{N-2}} dx = 0$$

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$(u_0, u_1) \mapsto G_+$ and $(u_0, u_1) \mapsto G_-$ are isometries between $\mathcal{H}$ and $L^2(\mathbb{R} \times S^{N-1})$.

Equirepartition of the energy

Theorem [TD, Kenig, Merle 2012]. *Assume N is odd. Let u_L be the solution of the linear wave equation with data (u_0, u_1). Then*

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{\{|x|\geq|t|\}} |\nabla_{t,x} u_L(t,x)|^2 \, dx = \int_{\mathbb{R}^N} |\nabla_{t,x} u_L(0,x)|^2 \, dx.$$

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More precisely:

- if $u_1 = 0$ then $G_+(\eta,\omega) = (-1)^{\frac{N+1}{2}} G_+(-\eta,-\omega)$.
- if $u_0 = 0$ then $G_+(\eta,\omega) = (-1)^{\frac{N-1}{2}} G_+(-\eta,-\omega)$,

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The lower bound ($\geq$) does not hold in even dimension ([Côte, Kenig, Schlag 2014]), even if a multiplicative constant is allowed.

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Nonradiative solution

Definition. *A (global) solution u of* (NLW) *is* *nonradiative* *when*

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Solitary waves are nonradiative.

We look for rigidity theorems of the form: u nonradiative $\Longrightarrow$ u is a solitary wave.

This is difficult: it implies in particular the nonexistence of pure multisolitons (typically false for integrable equations). See [Martel Merle 2017].

Rigidity theorem for small data

Theorem [TD, Kenig, Merle 2012]. *Assume N is odd. Then there exists* $\varepsilon_0 > 0$ *such that if u is a solution of* (NLW) *with:*

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$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{\{|x|\geq|t|\}} |\nabla_{t,x} u(t,x)|^2 \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla_{t,x} u(0,x)|^2 \, dx.$$

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Corollary [TD, Kenig, Merle 2012] *Soliton resolution for type II blow-up solutions holds with the additional assumption:*

$$\limsup_{t\to T_+} \|\vec{u}(t)\|^2_{\mathcal{H}} \leq \|W\|^2_{\dot{H}^1} + \varepsilon^2.$$

See works of [Krieger, Nakanishi, Schlag] for a complete description of the dynamics close to the ground state.

Improved exterior energy bound for radial solution

Let $R > 0$. Denote by $\mathcal{H}_R$ the space of radial functions in $(\dot{H}^1 \times L^2)(\{r > R\})$, and $\pi_R^{\perp}$ the orthogonal projection, in $\mathcal{H}_R$, on the orthogonal of $P_R = \mathrm{span}((1/r, 0))$.

Proposition. *Let u_L be a radial solution of the linear wave equation, in space dimension 3, with initial data (u_0, u_1). Then*

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{+\infty} (\partial_{t,r} u_L(t,r))^2 r^2 \, dr = \|\pi_R^{\perp}(u_0, u_1)\|_{\mathcal{H}_R}^2.$$

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Note that P_R is the intersection of $\bigcup_{k \geq 1} N\left(\Delta^k\right) \times \bigcup_{k \geq 1} N\left(\Delta^k\right)$ and $\mathcal{H}_R$.

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Generalization to other odd dimension, with P_R defined as above: [Kenig, Lawrie, Baoping Liu, Schlag 2015]. Note that:

$$\dim P_R = \frac{N-1}{2}$$

Rigidity theorem

Theorem. *Assume $N = 3$. If u is a radial non radiative solution, then $u = 0$ or there exist $\lambda > 0$, $\iota \in \{\pm 1\}$ such that $u(t,x) = \frac{\iota}{\lambda^{1/2}} W\left(\frac{x}{\lambda}\right)$.*

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First step of the proof: $\exists \ell \in \mathbb{R}$ such that

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Difficulty in higher odd dimensions due to the higher dimension of P_R!

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Lower bound for well-prepared initial data

Lemma. *Let $\gamma \in (0,1)$. There exists $\varepsilon = \varepsilon(\gamma) > 0$ with the following property. Let u_L be a solution of* (LW) *with initial data* (u_0, u_1) *such that*

$$\begin{cases} (u_0, u_1) \in \dot{H}^1 \times L^2 & \text{if } N \geq 3 \\ |\nabla u_0| \in L^2, u_1 \in L^2 \text{ and } u_0(x) = u_\infty \text{ for large } |x| & \text{if } N = 2 \end{cases}$$

(where $u_\infty \in \mathbb{R}$) and:

$$\|(\nabla u_0, u_1)\|_{L^2(B^c_{1+\varepsilon} \cup B_{1-\varepsilon})} + \|\not\partial u_0\|_{L^2} + \|\partial_r u_0 + u_1\|_{L^2} \leq \varepsilon \|(\nabla u_0, u_1)\|_{L^2}.$$

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Application: soliton resolution up to a sequence of times for (NLW) [TD, Jia, Kenig, Merle 2017]. Soliton resolution for small type II blow-up solutions for wave maps. [TD, Jia, Kenig, Merle 2016], using [Grinis 2016].

Wave maps

$$\text{(WM)} \qquad \begin{cases} \partial_t^2 u - \Delta u = \left(|\nabla u|^2 - |\partial_t u|^2\right) u, \quad x \in \mathbb{R}^2 \\ \vec{u}_{\restriction t=0} = (u_0, u_1), \quad u_0 \cdot u_1 = 0, \quad |u_0| = 1. \end{cases}$$

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Blow-up solutions [Krieger, Schlag and Tataru 2008], [Raphaël, Rodnianski 2012], [Jendrej 2016].

Well prepared initial data for wave maps

Theorem. *Let* $\gamma \in (0,1)$*. Then there exists* $\varepsilon = \varepsilon(\gamma) > 0$ *with the following property. Let u be a classical solution of* (WM) *with initial data* (u_0, u_1) *such that*

$$E_M(u_0, u_1) \leq \varepsilon$$

and

$$\|(\nabla u_0, u_1)\|_{L^2(B^c_{1+\varepsilon} \cup B_{1-\varepsilon})} + \|\not{\partial} u_0\|_{L^2} + \|\partial_r u_0 + u_1\|_{L^2} \leq \varepsilon \|(\nabla u_0, u_1)\|_{L^2}.$$

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Then for all $t \geq 0$*,*

$$\int_{|x| \geq \gamma + t} |\nabla_{x,t} u|^2(t,x)\, dx \geq \gamma \|(\nabla u_0,\ u_1)\|^2_{L^2}.$$

Small type II blow-up solutions for wave maps

Theorem. *Let u be a classical solution of* (WM) *such that $E_M(\vec{u}(0)) < E_M(S,0) + \epsilon_0^2$, blowing up in finite time T_+ at $x = 0$. Then there exists $\mathbf{p} \in \mathbb{R}^2$ such that $|\mathbf{p}| \ll 1$, $x(t) \in \mathbb{R}^2$, $\lambda(t) > 0$ with*

$$\lim_{t \to T_+} \frac{x(t)}{T_+ - t} = \mathbf{p}, \quad \lim_{t \to T_+} \frac{\lambda(t)}{T_+ - t} = 0,$$

and $(v_0,\, v_1) \in \mathcal{H} \cap C^\infty(\mathbb{R}^2 \setminus \{0\})$ with $(v_0 - u_\infty,\, v_1)$ compactly supported, such that

$$\text{(i)} \qquad \inf \left\{ \left\| \vec{u}(t) - (v_0,\, v_1) - (Q_{\mathbf{p}},\, \partial_t Q_{\mathbf{p}}) \right\|_{\mathcal{H}} \,:\, Q_{\mathbf{p}} \in \mathcal{M}_{\mathbf{p}} \right\} \xrightarrow[t \to T_+]{} 0,$$

$$\text{(ii)} \qquad \left\| (\nabla u(t), \partial_t u(t)) - (\nabla v_0,\, v_1) \right\|_{L^2\left(\mathbb{R}^2 \setminus B_{\lambda(t)}(x(t))\right)} \xrightarrow[t \to T_+]{} 0,$$

where $B_{\lambda(t)}(x(t)) = \left\{ x \in \mathbb{R}^2 \,:\, |x - x(t)| < \lambda(t) \right\}$, $\mathcal{M}_{\mathbf{p}}$ is the manifold of all transformations of $S_{\mathbf{p}}$ (by the group spanned by space translation, scaling, and $\mathbb{S}^2$ isometries).