Stable solitons of the cubic-quintic NLS with a delta-function potential

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The cubic-quintic NLS with a $\delta$-potential

We consider the nonlinear Schrödinger equation

$$i\psi_z + \psi_{xx} + \epsilon\delta(x)\psi + 2|\psi|^2\psi - |\psi|^4\psi = 0,$$  \hspace{1cm} (NLS)

for $\psi = \psi(x, z) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$.

This combination of nonlinearities is well known in nonlinear waveguides, including colloidal waveguides.

The delta-function models the interaction of a broad beam with a narrow trapping potential, with coupling constant $\epsilon > 0$.

The Cauchy problem for (NLS) is globally well-posed in $H^1(\mathbb{R})$.

(NLS) is the paraxial approximation of the nonlinear Helmholtz equation governing TE/TM modes in the waveguide.
Solitons

We look for standing waves of the form \( \psi(x, z) = e^{ikz} u(x) \), with \( k > 0 \) and a real-valued soliton profile \( u \in H^1(\mathbb{R}) \). This ansatz leads to the stationary equation

\[
    u'' - ku + \epsilon \delta(x) u + 2u^3 - u^5 = 0, \quad x \in \mathbb{R}.
\]  

(SNLS)

Orbital stability of standing waves relies on properties of the solutions of (SNLS) with respect to the wavenumber \( k \) (Vakhitov–Kolokolov ’73 … Grillakis–Shatah–Strauss ’87).

Our approach here is twofold:

- First determine all localised solutions \( u_k \) of (SNLS) explicitly.
- Then combine this information with spectral and bifurcation-theoretic properties to prove their stability.
A priori properties of solutions

Functions in $H^1(\mathbb{R})$ satisfying

$$u'' - ku + \epsilon \delta(x)u + 2u^3 - u^5 = 0 \quad \text{(SNLS)}$$

in the sense of distributions have the following properties:

(i) $u'' - ku + 2u^3 - u^5 = 0$, $x \neq 0$;

(ii) $\pm u > 0$ on $\mathbb{R}$;

(iii) $u$ is even on $\mathbb{R}$;

(iv) $u \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}) \cap H^2(\mathbb{R} \setminus \{0\})$;

(v) $u'(0^\pm) = \mp \frac{\epsilon}{2} u(0)$;

(vi) $u(x), u'(x) \to 0$ as $|x| \to \infty$. 
Brute force integration

The first integral of (SNLS) reads

$$(u')(x)^2 - ku^2(x) + u^4(x) - \frac{1}{3}u^6(x) = 0, \quad x \neq 0.$$  

Taking the limit $x \to 0^\pm$ yields

$$u^6(0) - 3u^4(0) + 3ku^2(0) - 3(u')^2(0) = 0$$
Brute force integration

The first integral of (SNLS) reads

\[(u')^2(x) - ku^2(x) + u^4(x) - \frac{1}{3} u^6(x) = 0, \quad x \neq 0.\]

Taking the limit \(x \to 0^\pm\) yields

\[u^6(0) - 3u^4(0) + 3ku^2(0) - 3\frac{\epsilon^2}{4} u^2(0) = 0\]
Brute force integration

The first integral of (SNLS) reads

\[(u')^2(x) - ku^2(x) + u^4(x) - \frac{1}{3} u^6(x) = 0, \quad x \neq 0.\]

Taking the limit \(x \to 0^\pm\) and dividing through by \(u^2(0)\) yields

\[u^4(0) - 3u^2(0) + 3(k - \frac{\epsilon^2}{4}) = 0,\]

and so

\[u^2_{\pm,k,\epsilon}(0) = \frac{3}{2} \left(1 \pm \sqrt{1 - \frac{4}{3} \left(k - \frac{\epsilon^2}{4}\right)}\right).\]

These numbers are real and positive if and only if

\[\frac{\epsilon^2}{4} < k \leq \frac{3}{4} + \frac{\epsilon^2}{4}.\]
To integrate further, we express $u'$ as

$$u'(x) = -\text{sgn}(x)u(x)\sqrt{\frac{1}{3}u^4(x) - u^2(x)} + k, \quad x \neq 0.$$ 

The positivity condition (PC) for the square root yields two different regimes:

(A) $\frac{\epsilon^2}{4} < k < \frac{3}{4}$:

$u^2_{\pm,k,\epsilon}(0)$ doesn’t satisfy (PC), there is only one soliton, $u_{-,k,\epsilon}$;

(B) $\frac{3}{4} < k < \frac{3}{4} + \frac{\epsilon^2}{4}$:

both $u^2_{\pm,k,\epsilon}(0)$ satisfy (PC), there are two solitons, $u_{\pm,k,\epsilon}$.

We note that regime (A) is void if $\epsilon \geq \sqrt{3}$, so we shall assume $0 < \epsilon < \sqrt{3}$ from now on. Regime (B) is void when $\epsilon = 0$. 
For $\epsilon = 0$, the solutions $u_k$ are given by (Pushkarov et al. '79)

$$u_k(x) = \sqrt{\frac{2k}{1 + \sqrt{1 - \frac{4k}{3} \cosh (2\sqrt{k}x)}}}, \quad 0 < k < \frac{3}{4}.$$
Two solutions coexist for each given wavenumber $k > \frac{3}{4}$, with a fold bifurcation occurring at $\overline{k}_\epsilon = \frac{3}{4} + \frac{\epsilon^2}{4}$. As we will see, the explicit formulas for the solitons are much more involved than for $\epsilon = 0$...
Figure: Soliton profiles for $\epsilon = 0.5 \cdot \sqrt{3}$. 
Figure: For $\epsilon = 0.9 \cdot \sqrt{3}$, plot of $\|u\|_{L^2}$ against $k \in \left(\frac{e^2}{4}, \frac{3}{4} + \frac{e^2}{4}\right]$. We will show that all solutions on the curve are orbitally stable.
Nonlinear optics:
Birnbaum–Malomed (2008)

Mathematical stability analysis:
Jeanjean–Fukuizumi (2008)
Explicit form of the solutions for $0 < \epsilon < \sqrt{3}$

For $\frac{\epsilon^2}{4} < k < \frac{3}{4}$: only one soliton for each $k$, given by

$$u_{-,k,\epsilon}(x) = \sqrt{2k} \frac{1 + \epsilon e^{\sqrt{1+\frac{\epsilon^2}{4k} - 1}} e^{2k|x|} + \frac{(1-\frac{4k}{3})(\sqrt{k} - \epsilon/2)}{\epsilon e^{\sqrt{1+\frac{\epsilon^2}{4k} - 1}}(1-\frac{4k}{3}) e^{-2\sqrt{k}|x|}}}{\sqrt{1 + \frac{\epsilon^2}{4k} e^{\sqrt{1+\frac{\epsilon^2}{4k} - 1}} e^{2\sqrt{k}|x|} + \frac{(1-\frac{4k}{3})(\sqrt{k} - \epsilon/2)}{\epsilon e^{\sqrt{1+\frac{\epsilon^2}{4k} - 1}}(1-\frac{4k}{3}) e^{-2\sqrt{k}|x|}}}.$$ 

At $k = \frac{3}{4}$, this reduces to

$$u_{-,3/4,\epsilon}(x) = \sqrt{\frac{3}{2}} \sqrt{\frac{1}{\frac{\epsilon}{\sqrt{3-\epsilon}}} e^{\sqrt{3}|x|}}.$$

For $\frac{3}{4} < k < \frac{3}{4} + \frac{\epsilon^2}{4}$: two solitons for each $k$, given by

$$u_{\pm,k,\epsilon}(x) = \frac{k}{2 \left( e^{\sqrt{k}(|x|-c)} + e^{-\sqrt{k}(|x|-c)} \right) \left( \left( 2 \sqrt{\frac{k}{3}} + 1 \right) e^{\sqrt{k}(|x|-c)} - \left( 2 \sqrt{\frac{k}{3}} - 1 \right) e^{-\sqrt{k}(|x|-c)} \right)},$$

where the integration constants $c = c_{\pm,k,\epsilon} \in \mathbb{R}$ can be determined from the values $u_{\pm,k,\epsilon}(0)$, which yields

$$e^{\sqrt{k}c_{-},k,\epsilon} = \sqrt{\frac{3 - \sqrt{3} \sqrt{3 + \epsilon^2 - 4k} + 2\epsilon \sqrt{k} - 4k}{-3 + \sqrt{3} \sqrt{3 + \epsilon^2 - 4k} + 2\sqrt{3} \sqrt{k - 2\sqrt{k} \sqrt{3 + \epsilon^2 - 4k}}}},$$

and

$$e^{\sqrt{k}c_{+},k,\epsilon} = \sqrt{\frac{-3 - \sqrt{3} \sqrt{3 + \epsilon^2 - 4k} - 2\epsilon \sqrt{k} + 4k}{3 + \sqrt{3} \sqrt{3 + \epsilon^2 - 4k} - 2\sqrt{3} \sqrt{k - 2\sqrt{k} \sqrt{3 + \epsilon^2 - 4k}}}}.$$
At the fold bifurcation point \((\bar{k}_\epsilon = \frac{3}{4} + \frac{\epsilon^2}{4})\) the solution takes the more tractable form

\[
\bar{u}_\epsilon(x) = \sqrt{\frac{3}{2}} \sqrt{\frac{3 + \epsilon^2}{3 + \epsilon^2 \cosh(\sqrt{3 + \epsilon^2} |x|) + \epsilon \sqrt{3 + \epsilon^2} \sinh(\sqrt{3 + \epsilon^2} |x|)}}.
\]

We will see that this expression is useful in the local spectral analysis at the fold bifurcation point \((\bar{k}_\epsilon, \bar{u}_\epsilon)\).
Bifurcation and spectral properties

We call lower curve, respectively upper curve, the sets

\[ S_{-, \epsilon} = \{(k, u_{-}, k, \epsilon) : k \in (\frac{\epsilon^2}{4}, k_{\epsilon})\}, \]

\[ S_{+, \epsilon} = \{(k, u_{+}, k, \epsilon) : k \in (\frac{3}{4}, k_{\epsilon})\}. \]

We then define

\[ S_{\epsilon} := S_{-}, \epsilon \cup \{(\bar{k}_{\epsilon}, \bar{u}_{\epsilon})\} \cup S_{+, \epsilon}. \]

We also let \( F_{\epsilon} : \mathbb{R} \times H^1(\mathbb{R}) \rightarrow H^{-1}(\mathbb{R}), \)

\[ F_{\epsilon}(k, u) = -u'' + ku - \epsilon \delta(x)u - 2u^3 + u^5, \]

so that (SNLS) reads \( F_{\epsilon}(k, u) = 0. \)
Theorem 1

(i) The set $S_\epsilon$ is a smooth curve in $\mathbb{R} \times H^1(\mathbb{R})$, and we have

$$\lim_{k \downarrow \frac{\epsilon^2}{4}} \| u_-, k, \epsilon \|_{H^1} = 0 \quad \text{and} \quad \lim_{k \downarrow \frac{3}{4}} \| u_+, k, \epsilon \|_{L^2} = \infty.$$ 

(ii) The linearised operator

$$D_u F_\epsilon(k, u) = -\frac{d^2}{dx^2} + k - \epsilon \delta(x) - 6u^2(x) + 5u^4(x),$$

is non-singular along $S_{\pm, \epsilon}$, and singular at $(k, u) = (\bar{k}_\epsilon, \bar{u}_\epsilon)$, with

$$\ker D_u F_\epsilon(\bar{k}_\epsilon, \bar{u}_\epsilon) = \text{span}\{\bar{u}'_\epsilon\}.$$ 

(iii) Furthermore, $D_u F_\epsilon(k, u)$ has a strictly positive continuous spectrum, and

- exactly one negative eigenvalue along $S_{-, \epsilon}$;
- no negative eigenvalues along $S_{+, \epsilon}$. 
Proof

- The bifurcations from $u = 0$ and from infinity follow by standard bifurcation theory.
- ODE arguments show that $D_u F_\epsilon(k, u)$ is an isomorphism for $k \neq \bar{k}_\epsilon$ and that $D_u F_\epsilon(\bar{k}_\epsilon, \bar{u}_\epsilon) = \text{span}\{|\bar{u}'_\epsilon|\}$.
- The smoothness of $S_{\pm, \epsilon}$ follows from the the implicit function theorem and the non-degeneracy of the solutions on $S_{\pm, \epsilon}$.
- One negative eigenvalue along $S_{-, \epsilon}$ follows from the case $\epsilon = 0$ by analytic perturbation theory.

Then it only remains to prove that:

- $S_{-, \epsilon}$ and $S_{+, \epsilon}$ meet smoothly at $(\bar{k}_\epsilon, \bar{u}_\epsilon)$.
- The first eigenvalue crosses zero with non-zero speed (and so doesn’t bounce back) as one passes through $(\bar{k}_\epsilon, \bar{u}_\epsilon)$. 
Local analysis at the fold

At the fold bifurcation point, parametrisation by $k$ breaks down. However, following Crandall and Rabinowitz ’73, the curve can be locally reparametrised about $(\bar{k}_\epsilon, \bar{u}_\epsilon)$ as a smooth curve

$$\{(k(s), u(s)) : s \in (-\eta, \eta)\} \subset \mathbb{R} \times H^1(\mathbb{R}) \quad (\eta > 0 \text{ small})$$

so that, at $s = 0$,

$$(k(0), u(0)) = (\bar{k}_\epsilon, \bar{u}_\epsilon) \quad \text{and} \quad (\dot{k}(0), \dot{u}(0)) = (0, |\bar{u}'_\epsilon|),$$

where $\ker D_u F_\epsilon(\bar{k}_\epsilon, \bar{u}_\epsilon) = \text{span}\{|\bar{u}'_\epsilon|\}$.

Now, the first eigenvalue $\mu(s)$ of $D_u F_\epsilon(k(s), u(s))$ satisfies $\mu(0) = 0$, and we only need to show that $\dot{\mu}(0) \neq 0$. 
To this aim, consider $\mu(s), v(s)$ the first eigenvalue, resp. eigenvector of $D_u F_\varepsilon(k(s), u(s)),\ s \in (-\eta, \eta)$.

At $s = 0$, we have $D_u F_\varepsilon(k(0), u(0)) = D_u F_\varepsilon(k_\varepsilon, u_\varepsilon)$, so

$$\mu(0) = 0, \quad v(0) = |\bar{u}_\varepsilon'|,$$

i.e.

$$D_u F_\varepsilon(k_\varepsilon, u_\varepsilon)|\bar{u}_\varepsilon'| = 0.$$

Furthermore, $D_u F_\varepsilon(k(s), u(s))v(s) = \mu(s)v(s)$ reads, for $x \neq 0$,

$$-v(s)'' + k(s)v(s) - [6u(s)^2 - 5u(s)^4]v(s) = \mu(s)v(s).$$

Recalling that

$$k(0) = \bar{k}_\varepsilon, \quad u(0) = \bar{u}_\varepsilon, \quad \dot{k}(0) = 0, \quad \dot{u}(0) = |\bar{u}_\varepsilon'|,$$

differentiation with respect to $s$ at $s = 0$ yields

$$-\dot{v}(0)'' + \bar{k}_\varepsilon \dot{v}(0) - [12\bar{u}_\varepsilon - 20\bar{u}_\varepsilon^3]|\bar{u}_\varepsilon'|^2 - [6\bar{u}_\varepsilon^2 - 5\bar{u}_\varepsilon^4]\dot{v}(0) = \dot{\mu}(0)|\bar{u}_\varepsilon'|.$$
We can then eliminate $\dot{\nu}(0)$ by combining this equation multiplied by $|\overline{u}'_\epsilon|$ with the equation $D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon)|\overline{u}'_\epsilon| = 0$ multiplied by $\dot{\nu}(0)$, and integrating by parts. This yields

$$\dot{\mu}(0) = \frac{4 \int_{\mathbb{R}} (5\overline{u}^2_\epsilon - 3) \overline{u}_\epsilon |\overline{u}'_\epsilon|^3}{\int_{\mathbb{R}} |\overline{u}'_\epsilon|^2}.$$ 

Thanks to the explicit formulas for $\overline{u}_\epsilon$ and $|\overline{u}'_\epsilon|$, we can plot the function $\epsilon \mapsto \int_{0}^{\infty} (5\overline{u}^2_\epsilon - 3) \overline{u}_\epsilon |\overline{u}'_\epsilon|^3$:

**Figure:** The graph of $\epsilon \mapsto \int_{0}^{\infty} (5\overline{u}^2_\epsilon - 3) \overline{u}_\epsilon |\overline{u}'_\epsilon|^3$. 

□
Stability

Due to the $U(1)$-invariance of (NLS), the appropriate notion of stability in this context is that of orbital stability.

**Definition**

We say that the standing wave $\psi_k(x, z) = e^{ikz} u_k(x)$ is orbitally stable if

$$\text{for all } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that}$$

for any solution $\varphi(x, z)$ of (NLS) with initial data $\varphi(\cdot, 0) \in H^1(\mathbb{R})$ there holds

$$\| \varphi(\cdot, 0) - u_k \|_{H^1} \leq \delta \implies \inf_{\theta \in \mathbb{R}} \| \varphi(\cdot, z) - e^{i\theta} u_k \|_{H^1} \leq \varepsilon \text{ for all } z \geq 0.$$
Theorem 2

*The whole curve $S_\epsilon$ consists of orbitally stable standing waves.*

**Proof.**

We use the general theory of Grillakis–Shatah–Strauss:

(I) We know that the spectrum of $D_u F_\epsilon(k, u_+, k, \epsilon)$ is strictly positive, for all $k \in \left(\frac{3}{4}, \bar{k}_\epsilon\right)$, so the upper curve is stable.

(II) We also know that, for all $k \in \left(\frac{\epsilon^2}{4}, \bar{k}_\epsilon\right)$, $D_u F_\epsilon(k, u_-, k, \epsilon)$ has exactly one simple negative eigenvalue, is non-singular, and the rest of its spectrum is strictly positive.

Hence, to complete the proof, we only need to verify the slope condition:

$$\frac{d}{dk} \|u_-, k, \epsilon\|_{L^2}^2 > 0 \quad \forall k \in \left(\frac{\epsilon^2}{4}, \bar{k}_\epsilon\right).$$
Firstly, for \( k \in (\frac{3}{4}, \overline{k}_\epsilon) \), it follows from the expression

\[
2 \sqrt{\frac{k}{\left( e^{\sqrt{k}(|x|-c)} + e^{-\sqrt{k}(|x|-c)} \right) \left( (2\sqrt{\frac{k}{3}} + 1)e^{\sqrt{k}(|x|-c)} - \left(2\sqrt{\frac{k}{3}} - 1\right)e^{-\sqrt{k}(|x|-c)} \right)}}
\]

that

\[
\frac{d}{dk} \| u_{-,k,\epsilon} \|_{L^2}^2 = \frac{2\sqrt{3\epsilon}}{\sqrt{3+\epsilon^2-4k}} - \frac{3}{\sqrt{k}} > 0.
\]

\text{N.B.} \text{ Similarly,}

\[
\frac{d}{dk} \| u_{+,k,\epsilon} \|_{L^2}^2 = -\frac{2\sqrt{3\epsilon}}{\sqrt{3+\epsilon^2-4k}} + \frac{3}{\sqrt{k}} < 0 \quad \forall k \in (\frac{3}{4}, \overline{k}_\epsilon).
\]
For $k < 3/4$, a straightforward (but painful) calculation using

$$u_{-, k, \epsilon}(x) = \sqrt{\frac{2k}{1 + \frac{\epsilon + \epsilon \sqrt{1 + (4k/\epsilon^2 - 1)(1 - 4k/3)}}{4(\sqrt{k} - \epsilon/2)} e^{2\sqrt{k}|x|} + \frac{(1 - 4k/3)(\sqrt{k} - \epsilon/2)}{\epsilon + \epsilon \sqrt{1 + (4k/\epsilon^2 - 1)(1 - 4k/3)}} e^{-2\sqrt{k}|x|}}$$

shows that

$$\|u_{-, k, \epsilon}\|_{L^2}^2 = \sqrt{3 \log \varphi_{\epsilon}(k)}$$

where

$$\varphi_{\epsilon}(k) := \frac{\sqrt{3\epsilon} + \sqrt{3\epsilon^2 + (4k - \epsilon^2)(3 - 4k)} + (\sqrt{3} + 2\sqrt{k})(2\sqrt{k} - \epsilon)}{\sqrt{3\epsilon} + \sqrt{3\epsilon^2 + (4k - \epsilon^2)(3 - 4k)} + (\sqrt{3} - 2\sqrt{k})(2\sqrt{k} - \epsilon)}.$$
Finally, \[
\frac{d}{dk} \varphi_\epsilon(k) =
\]
\[
8\sqrt{k} \frac{\sqrt{3} \sqrt{3\epsilon^2} + (4k - \epsilon^2)(3 - 4k) + 2\sqrt{k}(3 + \epsilon^2 - 2\epsilon\sqrt{k})}{\sqrt{3\epsilon^2 + (4k - \epsilon^2)(3 - 4k)}\left(\sqrt{3\epsilon} + \sqrt{3\epsilon^2 + (4k - \epsilon^2)(3 - 4k)} + (\sqrt{3} - 2\sqrt{k})(2\sqrt{k} - \epsilon)\right)^2}
\]
which is strictly positive since
\[
k < \frac{3}{4} \implies 3 + \epsilon^2 - 2\epsilon\sqrt{k} > (\sqrt{3} - \epsilon)^2 + \sqrt{3}\epsilon > 0.
\]

Thank you!