Stable solitons of the cubic-quintic NLS with a delta-function potential

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Toulouse, October 3, 2018

The cubic-quintic NLS with a δ-potential

We consider the nonlinear Schrödinger equation

$$i\psi_z + \psi_{xx} + \epsilon\delta(x)\psi + 2|\psi|^2\psi - |\psi|^4\psi = 0, \tag{NLS}$$

for $\psi = \psi(x, z) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.

This combination of nonlinearities is well known in nonlinear waveguides, including colloidal waveguides.

The delta-function models the interaction of a broad beam with a narrow trapping potential, with coupling constant $\epsilon > 0$.

The Cauchy problem for (NLS) is globally well-posed in $H^1(\mathbb{R})$.

(NLS) is the paraxial approximation of the nonlinear Helmholtz equation governing TE/TM modes in the waveguide.

Solitons

We look for standing waves of the form $\psi(x,z) = e^{ikz}u(x)$, with $k > 0$ and a real-valued soliton profile $u \in H^1(\mathbb{R})$. This ansatz leads to the stationary equation

$$u'' - ku + \epsilon\delta(x)u + 2u^3 - u^5 = 0, \quad x \in \mathbb{R}. \tag{SNLS}$$

Orbital stability of standing waves relies on properties of the solutions of (SNLS) with respect to the wavenumber k (Vakhitov–Kolokolov '73 ... Grillakis–Shatah–Strauss '87).

Our approach here is twofold:

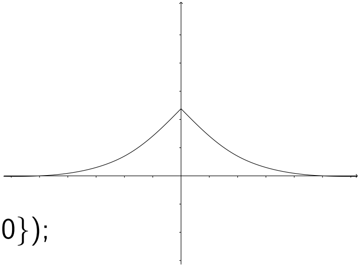
- First determine all localised solutions u_k of (SNLS) explicitly.
- Then combine this information with spectral and bifurcation-theoretic properties to prove their stability.

A priori properties of solutions

Functions in $H^1(\mathbb{R})$ satisfying

$$u'' - ku + \epsilon\delta(x)u + 2u^3 - u^5 = 0 \qquad \text{(SNLS)}$$

in the sense of distributions have the following properties:



(i) $u'' - ku + 2u^3 - u^5 = 0, \ x \neq 0$;

(ii) $\pm u > 0$ on $\mathbb{R}$;

(iii) u is even on $\mathbb{R}$;

(iv) $u \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}) \cap H^2(\mathbb{R} \setminus \{0\})$;

(v) $u'(0^{\pm}) = \mp \frac{\epsilon}{2} u(0)$;

(vi) $u(x), u'(x) \to 0$ as $|x| \to \infty$.

Brute force integration

The first integral of (SNLS) reads

$$(u')^2(x) - ku^2(x) + u^4(x) - \frac{1}{3}u^6(x) = 0, \quad x \neq 0.$$

Taking the limit $x \to 0^{\pm}$ yields

$$u^6(0) - 3u^4(0) + 3ku^2(0) - 3(u')^2(0) = 0$$

Brute force integration

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Taking the limit $x \to 0^{\pm}$ yields

$$u^6(0) - 3u^4(0) + 3ku^2(0) - 3\frac{\epsilon^2}{4}u^2(0) = 0$$

Brute force integration

The first integral of (SNLS) reads

$$(u')^2(x) - ku^2(x) + u^4(x) - \tfrac{1}{3}u^6(x) = 0, \quad x \neq 0.$$

Taking the limit $x \to 0^{\pm}$ and dividing through by $u^2(0)$ yields

$$u^4(0) - 3u^2(0) + 3(k - \tfrac{\epsilon^2}{4}) = 0,$$

and so

$$u^2_{\pm,k,\epsilon}(0) = \tfrac{3}{2}\left(1 \pm \sqrt{1 - \tfrac{4}{3}\left(k - \tfrac{\epsilon^2}{4}\right)}\right).$$

These numbers are real and positive if and only if

$$\tfrac{\epsilon^2}{4} < k \leqslant \tfrac{3}{4} + \tfrac{\epsilon^2}{4}.$$

To integrate further, we express u' as

$$u'(x) = -\text{sgn}(x)u(x)\sqrt{\tfrac{1}{3}u^4(x) - u^2(x) + k}, \quad x \neq 0.$$

The positivity condition (PC) for the square root yields two different regimes:

(A) $\frac{\epsilon^2}{4} < k < \frac{3}{4}$:
$u^2_{+,k,\epsilon}(0)$ doesn't satisfy (PC), there is only one soliton, $u_{-,k,\epsilon}$;

(B) $\frac{3}{4} < k < \frac{3}{4} + \frac{\epsilon^2}{4}$:
both $u^2_{\pm,k,\epsilon}(0)$ satisfy (PC), there are two solitons, $u_{\pm,k,\epsilon}$.

We note that regime (A) is void if $\epsilon \geqslant \sqrt{3}$, so we shall assume $0 < \epsilon < \sqrt{3}$ from now on. Regime (B) is void when $\epsilon = 0$.

$\epsilon = 0$

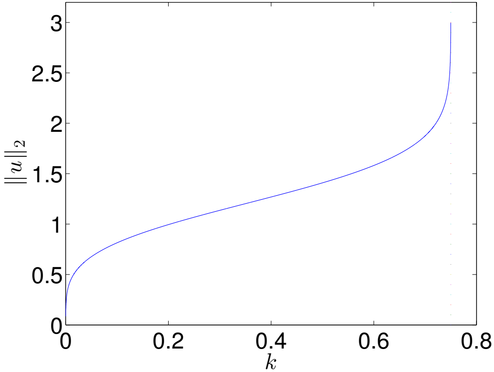

Figure: For $\epsilon = 0$, plot of $\|u\|_{L^2}$ against $k \in (0, \frac{3}{4})$.

For $\epsilon = 0$ the solutions u_k are given by (Pushkarov *et al.* '79)

$$u_k(x) = \sqrt{\frac{2k}{1 + \sqrt{1 - \frac{4k}{3}} \cosh\left(2\sqrt{k}x\right)}}, \quad 0 < k < \tfrac{3}{4}.$$

$0 < \epsilon < \sqrt{3}$

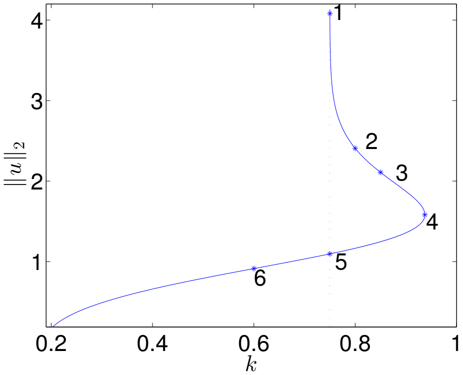

Figure: For $\epsilon = 0.5 \cdot \sqrt{3}$, plot of $\|u\|_{L^2}$ against $k \in (\frac{\epsilon^2}{4}, \frac{3}{4} + \frac{\epsilon^2}{4}]$.

Two solutions coexist for each given wavenumber $k > \frac{3}{4}$, with a fold bifurcation occuring at $\overline{k}_\epsilon = \frac{3}{4} + \frac{\epsilon^2}{4}$. As we will see, the explicit formulas for the solitons are much more involved than for $\epsilon = 0$...

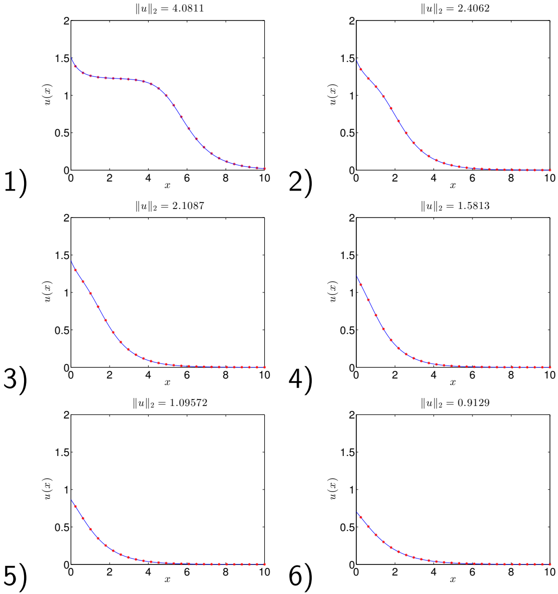

Figure: Soliton profiles for $\epsilon = 0.5 \cdot \sqrt{3}$.

$\epsilon \approx \sqrt{3}$

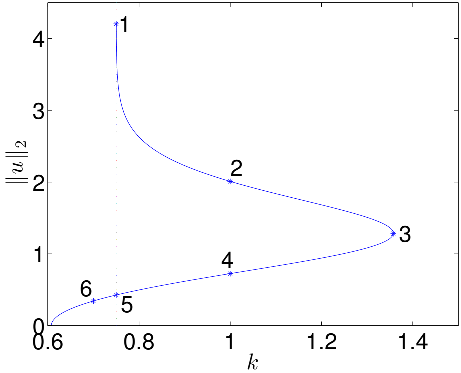

Figure: For $\epsilon = 0.9 \cdot \sqrt{3}$, plot of $\|u\|_{L^2}$ against $k \in (\frac{\epsilon^2}{4}, \frac{3}{4} + \frac{\epsilon^2}{4}]$.

We will show that all solutions on the curve are orbitally stable.

Literature

Nonlinear optics:

Gisin–Driben–Malomed (2004)

Birnbaum–Malomed (2008)

Mathematical stability analysis:

Fukuizumi–Ohta–Ozawa (2008)

Jeanjean–Fukuizumi (2008)

Le Coz–Fukuizumi–Fibich–Ksherim–Sivan (2008)

Pava–Melo (2018, arXiv preprint)

Explicit form of the solutions for $0 < \epsilon < \sqrt{3}$

For $\frac{\epsilon^2}{4} < k < \frac{3}{4}$: only one soliton for each k, given by

$$u_{-,k,\epsilon}(x) = \sqrt{\frac{2k}{1 + \frac{\epsilon+\epsilon\sqrt{1+(4k/\epsilon^2-1)(1-4k/3)}}{4(\sqrt{k}-\epsilon/2)} e^{2\sqrt{k}|x|} + \frac{(1-4k/3)(\sqrt{k}-\epsilon/2)}{\epsilon+\epsilon\sqrt{1+(4k/\epsilon^2-1)(1-4k/3)}} e^{-2\sqrt{k}|x|}}}.$$

At $k = 3/4$, this reduces to

$$u_{-,3/4,\epsilon}(x) = \sqrt{\frac{3}{2}} \sqrt{\frac{1}{1 + \frac{\epsilon}{\sqrt{3}-\epsilon} e^{\sqrt{3}|x|}}}.$$

For $\frac{3}{4} < k < \frac{3}{4} + \frac{\epsilon^2}{4}$: two solitons for each k, given by

$$u_{\pm,k,\epsilon}(x) =$$

$$2\sqrt{\frac{k}{\left(e^{\sqrt{k}(|x|-c)} + e^{-\sqrt{k}(|x|-c)}\right)\left(\left(2\sqrt{\frac{k}{3}}+1\right)e^{\sqrt{k}(|x|-c)} - \left(2\sqrt{\frac{k}{3}}-1\right)e^{-\sqrt{k}(|x|-c)}\right)}},$$

where the integration constants $c = c_{\pm,k,\epsilon} \in \mathbb{R}$ can be determined from the values $u_{\pm,k,\epsilon}(0)$, which yields

$$e^{\sqrt{k}c_{-,k,\epsilon}} = \sqrt{\frac{3 - \sqrt{3}\sqrt{3+\epsilon^2-4k} + 2\epsilon\sqrt{k} - 4k}{-3 + \sqrt{3}\sqrt{3+\epsilon^2-4k} + 2\sqrt{3}\sqrt{k} - 2\sqrt{k}\sqrt{3+\epsilon^2-4k}}}$$

and

$$e^{\sqrt{k}c_{+,k,\epsilon}} = \sqrt{\frac{-3 - \sqrt{3}\sqrt{3+\epsilon^2-4k} - 2\epsilon\sqrt{k} + 4k}{3 + \sqrt{3}\sqrt{3+\epsilon^2-4k} - 2\sqrt{3}\sqrt{k} - 2\sqrt{k}\sqrt{3+\epsilon^2-4k}}}.$$

At the fold bifurcation point ($\overline{k}_\epsilon = \frac{3}{4} + \frac{\epsilon^2}{4}$) the solution takes the more tractable form

$$\overline{u}_\epsilon(x) = \sqrt{\frac{3}{2}}\sqrt{\frac{3+\epsilon^2}{3+\epsilon^2\cosh(\sqrt{3+\epsilon^2}|x|)+\epsilon\sqrt{3+\epsilon^2}\sinh(\sqrt{3+\epsilon^2}|x|)}}.$$

We will see that this expression is useful in the local spectral analysis at the fold bifurcation point $(\overline{k}_\epsilon, \overline{u}_\epsilon)$.

Bifurcation and spectral properties

We call lower curve, respectively upper curve, the sets

$$\mathcal{S}_{-,\epsilon} = \left\{(k, u_{-,k,\epsilon}) : k \in (\tfrac{\epsilon^2}{4}, \overline{k}_\epsilon)\right\},$$

$$\mathcal{S}_{+,\epsilon} = \left\{(k, u_{+,k,\epsilon}) : k \in (\tfrac{3}{4}, \overline{k}_\epsilon)\right\}.$$

We then define

$$\mathcal{S}_\epsilon := \mathcal{S}_{-,\epsilon} \cup \{(\overline{k}_\epsilon, \overline{u}_\epsilon)\} \cup \mathcal{S}_{+,\epsilon}.$$

We also let $F_\epsilon : \mathbb{R} \times H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$,

$$F_\epsilon(k, u) = -u'' + ku - \epsilon\delta(x)u - 2u^3 + u^5,$$

so that (SNLS) reads $F_\epsilon(k, u) = 0$.

Theorem 1

(i) *The set $\mathcal{S}_\epsilon$ is a smooth curve in $\mathbb{R} \times H^1(\mathbb{R})$, and we have*

$$\lim_{k \downarrow \frac{\epsilon^2}{4}} \|u_{-,k,\epsilon}\|_{H^1} = 0 \quad \textit{and} \quad \lim_{k \downarrow \frac{3}{4}} \|u_{+,k,\epsilon}\|_{L^2} = \infty.$$

(ii) *The linearised operator*

$$D_u F_\epsilon(k,u) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k - \epsilon\delta(x) - 6u^2(x) + 5u^4(x),$$

is non-singular along $\mathcal{S}_{\pm,\epsilon}$, and singular at $(k,u) = (\overline{k}_\epsilon, \overline{u}_\epsilon)$, with

$$\ker D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon) = \mathrm{span}\{|\overline{u}_\epsilon'|\}.$$

(iii) *Furthermore, $D_u F_\epsilon(k,u)$ has a strictly positive continuous spectrum, and*

- *exactly one negative eigenvalue along $\mathcal{S}_{-,\epsilon}$;*
- *no negative eigenvalues along $\mathcal{S}_{+,\epsilon}$.*

Proof

- The bifurcations from $u = 0$ and from infinity follow by standard bifurcation theory.
- ODE arguments show that $D_u F_\epsilon(k, u)$ is an isomorphism for $k \neq \overline{k}_\epsilon$ and that $D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon) = \mathrm{span}\{|\overline{u}'_\epsilon|\}$.
- The smoothness of $\mathcal{S}_{\pm,\epsilon}$ follows from the the implicit function theorem and the non-degeneracy of the solutions on $\mathcal{S}_{\pm,\epsilon}$.
- One negative eigenvalue along $\mathcal{S}_{-,\epsilon}$ follows from the case $\epsilon = 0$ by analytic perturbation theory.

Then it only remains to prove that:

- $\mathcal{S}_{-,\epsilon}$ and $\mathcal{S}_{+,\epsilon}$ meet smoothly at $(\overline{k}_\epsilon, \overline{u}_\epsilon)$.
- The first eigenvalue crosses zero with non-zero speed (and so doesn't bounce back) as one passes through $(\overline{k}_\epsilon, \overline{u}_\epsilon)$.

Local analysis at the fold

At the fold bifurcation point, parametrisation by k breaks down. However, following Crandall and Rabinowitz '73, the curve can be locally reparametrised about $(\overline{k}_\epsilon, \overline{u}_\epsilon)$ as a smooth curve

$$\big\{(k(s), u(s)) : s \in (-\eta, \eta)\big\} \subset \mathbb{R} \times H^1(\mathbb{R}) \quad (\eta > 0 \text{ small})$$

so that, at $s = 0$,

$$(k(0), u(0)) = (\overline{k}_\epsilon, \overline{u}_\epsilon) \quad \text{and} \quad (\dot{k}(0), \dot{u}(0)) = (0, |\overline{u}'_\epsilon|),$$

where $\ker D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon) = \operatorname{span}\{|\overline{u}'_\epsilon|\}$.

Now, the first eigenvalue $\mu(s)$ of $D_u F_\epsilon(k(s), u(s))$ satisfies $\mu(0) = 0$, and we only need to show that $\dot{\mu}(0) \neq 0$.

To this aim, consider $\mu(s), v(s)$ the first eigenvalue, resp. eigenvector of $D_u F_\epsilon(k(s), u(s))$, $s \in (-\eta, \eta)$.

At $s = 0$, we have $D_u F_\epsilon(k(0), u(0)) = D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon)$, so

$$\mu(0) = 0, \quad v(0) = |\overline{u}_\epsilon'|, \quad \text{i.e.} \quad D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon)|\overline{u}_\epsilon'| = 0.$$

Furthermore, $D_u F_\epsilon(k(s), u(s))v(s) = \mu(s)v(s)$ reads, for $x \neq 0$,

$$-v(s)'' + k(s)v(s) - \left[6u(s)^2 - 5u(s)^4\right] v(s) = \mu(s)v(s).$$

Recalling that

$$k(0) = \overline{k}_\epsilon, \quad u(0) = \overline{u}_\epsilon, \quad \dot{k}(0) = 0, \quad \dot{u}(0) = |\overline{u}_\epsilon'|,$$

differentiation with respect to s at $s = 0$ yields

$$-\dot{v}(0)'' + \overline{k}_\epsilon \dot{v}(0) - \left[12\overline{u}_\epsilon - 20\overline{u}_\epsilon^3\right] |\overline{u}_\epsilon'|^2 - \left[6\overline{u}_\epsilon^2 - 5\overline{u}_\epsilon^4\right] \dot{v}(0) = \dot{\mu}(0)|\overline{u}_\epsilon'|.$$

We can then eliminate $\dot{v}(0)$ by combining this equation multiplied by $|\overline{u}_\epsilon'|$ with the equation $D_u F_\epsilon(\overline{k}_\epsilon, \overline{u}_\epsilon)|\overline{u}_\epsilon'| = 0$ multiplied by $\dot{v}(0)$, and integrating by parts. This yields

$$\dot{\mu}(0) = \frac{4\int_{\mathbb{R}} \left(5\overline{u}_\epsilon^2 - 3\right)\overline{u}_\epsilon|\overline{u}_\epsilon'|^3}{\int_{\mathbb{R}} |\overline{u}_\epsilon'|^2}.$$

Thanks to the explicit formulas for $\overline{u}_\epsilon$ and $|\overline{u}_\epsilon'|$, we can plot the function $\epsilon \mapsto \int_0^\infty \left(5\overline{u}_\epsilon^2 - 3\right)\overline{u}_\epsilon|\overline{u}_\epsilon'|^3$:

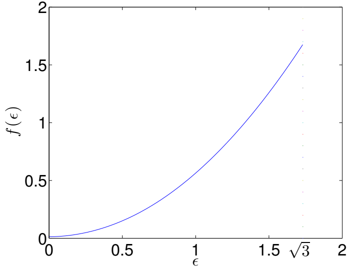

Figure: The graph of $\epsilon \mapsto \int_0^\infty \left(5\overline{u}_\epsilon^2 - 3\right)\overline{u}_\epsilon|\overline{u}_\epsilon'|^3$.

$\square$

Stability

Due to the $U(1)$-invariance of (NLS), the appropriate notion of stability in this context is that of orbital stability.

Definition

We say that the standing wave $\psi_k(x, z) = e^{ikz} u_k(x)$ is orbitally stable if

for all $\varepsilon > 0$ there exists $\delta > 0$ such that

for any solution $\varphi(x, z)$ of (NLS) with initial data $\varphi(\cdot, 0) \in H^1(\mathbb{R})$ there holds

$$\|\varphi(\cdot, 0) - u_k\|_{H^1} \leqslant \delta \implies \inf_{\theta \in \mathbb{R}} \|\varphi(\cdot, z) - e^{i\theta} u_k\|_{H^1} \leqslant \varepsilon \quad \text{for all } z \geqslant 0.$$

Theorem 2

The whole curve $\mathcal{S}_\epsilon$ consists of orbitally stable standing waves.

Proof.

We use the general theory of Grillakis–Shatah–Strauss:

(I) We know that the spectrum of $D_u F_\epsilon(k, u_{+,k,\epsilon})$ is strictly positive, for all $k \in (\frac{3}{4}, \overline{k}_\epsilon)$, so the upper curve is stable.

(II) We also know that, for all $k \in (\frac{\epsilon^2}{4}, \overline{k}_\epsilon)$, $D_u F_\epsilon(k, u_{-,k,\epsilon})$ has exactly one simple negative eigenvalue, is non-singular, and the rest of its spectrum is strictly positive.

Hence, to complete the proof, we only need to verify the slope condition:

$$\frac{\mathrm{d}}{\mathrm{d}k}\|u_{-,k,\epsilon}\|_{L^2}^2 > 0 \quad \forall k \in (\tfrac{\epsilon^2}{4}, \overline{k}_\epsilon).$$

Firstly, for $k \in (\frac{3}{4}, \overline{k}_\epsilon)$, it follows from the expression

$$u_{-,k,\epsilon}(x) =$$
$$2\sqrt{\frac{k}{\left(e^{\sqrt{k}(|x|-c)} + e^{-\sqrt{k}(|x|-c)}\right)\left(\left(2\sqrt{\frac{k}{3}}+1\right)e^{\sqrt{k}(|x|-c)} - \left(2\sqrt{\frac{k}{3}}-1\right)e^{-\sqrt{k}(|x|-c)}\right)}},$$

that

$$\frac{d}{dk}\|u_{-,k,\epsilon}\|_{L^2}^2 = \frac{\frac{2\sqrt{3}\epsilon}{\sqrt{3+\epsilon^2-4k}} - \frac{3}{\sqrt{k}}}{4k-3} > 0.$$

N.B. Similarly,

$$\frac{d}{dk}\|u_{+,k,\epsilon}\|_{L^2}^2 = -\frac{\frac{2\sqrt{3}\epsilon}{\sqrt{3+\epsilon^2-4k}} + \frac{3}{\sqrt{k}}}{4k-3} < 0 \quad \forall k \in (\tfrac{3}{4}, \overline{k}_\epsilon).$$

For $k < 3/4$, a straightforward (but painful) calculation using

$$u_{-,k,\epsilon}(x) = \sqrt{\frac{2k}{1 + \frac{\epsilon+\epsilon\sqrt{1+(4k/\epsilon^2-1)(1-4k/3)}}{4(\sqrt{k}-\epsilon/2)} e^{2\sqrt{k}|x|} + \frac{(1-4k/3)(\sqrt{k}-\epsilon/2)}{\epsilon+\epsilon\sqrt{1+(4k/\epsilon^2-1)(1-4k/3)}} e^{-2\sqrt{k}|x|}}}$$

shows that

$$\|u_{-,k,\epsilon}\|_{L^2}^2 = \sqrt{3}\log\varphi_\epsilon(k)$$

where

$$\varphi_\epsilon(k) := \frac{\sqrt{3}\epsilon + \sqrt{3\epsilon^2 + (4k-\epsilon^2)(3-4k)} + \left(\sqrt{3} + 2\sqrt{k}\right)\left(2\sqrt{k} - \epsilon\right)}{\sqrt{3}\epsilon + \sqrt{3\epsilon^2 + (4k-\epsilon^2)(3-4k)} + \left(\sqrt{3} - 2\sqrt{k}\right)\left(2\sqrt{k} - \epsilon\right)}.$$

Finally, $\quad \dfrac{\mathrm{d}}{\mathrm{d}k}\varphi_\epsilon(k) =$

$$8\sqrt{k}\frac{\sqrt{3}\sqrt{3\epsilon^2+(4k-\epsilon^2)(3-4k)}+2\sqrt{k}\left(3+\epsilon^2-2\epsilon\sqrt{k}\right)}{\sqrt{3\epsilon^2+(4k-\epsilon^2)(3-4k)}\left(\sqrt{3}\epsilon+\sqrt{3\epsilon^2+(4k-\epsilon^2)(3-4k)}+\left(\sqrt{3}-2\sqrt{k}\right)\left(2\sqrt{k}-\epsilon\right)\right)^2}$$

which is strictly positive since

$$k<\frac{3}{4} \implies 3+\epsilon^2-2\epsilon\sqrt{k} > \left(\sqrt{3}-\epsilon\right)^2+\sqrt{3}\epsilon > 0. \qquad \square$$

Thank you!