Semi-stabilization for the semilinear damped wave equation

Romain JOLY
Université Grenoble Alpes

Joint work with Camille Laurent, CNRS-Paris VI

Toulouse, 2 octobre 2018
The semilinear damped wave equation

\[ \partial_{tt}^2 u + \gamma(x) \partial_t u = \Delta u - f(x, u) \]

- \( \Omega \) is a smooth compact manifold of dimension \( d = 2 \) with Dirichlet boundary conditions.
- the damping \( \gamma \) is in \( L^\infty(\Omega) \), \( \gamma(x) \geq 0 \)
- \( f \) is smooth and of degree \( p \)
  \[ |f(x, u)| + |f'(x, u)| \leq C(1 + |u|)^p \]
  \[ |f'(x, u)| \leq C(1 + |u|)^{p-1} \]
- \( f \) is asymptotically of the sign of \( u \):
  \[ \forall |u| \geq R \ , \ f(x, u)u \geq 0 \]
\[ \partial_{tt}^2 u + \gamma(x) \partial_t u = \Delta u - f(x, u) \]

Set \( X = H^1_0(\Omega) \times L^2(\Omega) \) and 

\[ U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \quad A = \begin{pmatrix} 0 & \text{Id} \\ \Delta & -\gamma(x) \end{pmatrix} \quad F(U) = \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} \]

\( \Rightarrow \) \( e^{At} \) is a dissipative semigroup on \( X \).

\( \Rightarrow \) Since \( f \) is of degree \( p < \infty \) and \( \Omega \) is of dimension \( d = 2 \), 

\( F : X \rightarrow X \) is defined and Lipschitz on the bounded sets.

We consider in \( X \) the equation 

\[ \partial_t U = AU + F(U) \quad U(t = 0) = U_0 \in X \]
The gradient dynamics

Set \( V(x, u) = \int_0^u f(x, \xi) \, d\xi \). The energy

\[
\mathcal{E}(U) = \int_\Omega \frac{1}{2} (|\nabla u|^2 + |\partial_t u|^2) + V(x, u) \, dx
\]

is non-increasing along the trajectories since

\[
\partial_t \mathcal{E}(U(t)) = -\int_\Omega \gamma(x)|\partial_t u|^2 \, dx
\]

\( \Rightarrow \) **Global existence of solutions**
The linear equation is **dissipative** and any solution goes to zero

\[ \|e^{At} U\|_X \xrightarrow{t \to +\infty} 0. \]

Do we still have stabilization of the nonlinear problem?

We expect gradient-like dynamics in $X$: there exists a compact global attractor and any trajectory converges to the set of equilibrium points.
Motivations

Related problems:

- Control and stabilization problems.
- Does a linear dissipative (dispersive?) behaviour remains after addition of a nonlinearity?
- Convergence to equilibria/travelling fronts in nonlinear PDEs

\[ \gamma > 0 \]
A historic result

1. A historic result

2. A standard extension

3. The disk with two holes

4. The disk with three holes

5. Conclusion
A historic result

\[ \partial_{tt} u + \gamma(x) \partial_t u = \Delta u - f(x, u) \]

Assume:

- \( \gamma(x) \geq \alpha > 0 \) in \( \Omega \)
- \( f \) is of degree \( p < \infty \)
- \( f \) is asymptotically of the sign of \( u \)


With the above assumptions, the dynamics of the damped wave equation admit a **compact global attractor** \( A \). Moreover, it is gradient-like and any trajectory converges to the sets of equilibrium points.
Step 1: the trajectories are bounded.

If $f$ is asymptotically of the sign of $u$ and of degree $p$, then the energy is well defined and bounded on bounded sets.

$$\frac{1}{2} \| U \|_X^2 + \min V \leq \int_{\Omega} \frac{1}{2} (|\nabla u|^2 + |\partial_t u|^2) + V(x, u) \, dx \leq K(\| U \|_X).$$

Since $\mathcal{E}$ is non-increasing, the trajectories of bounded sets are bounded.
Step 2: the asymptotic compactness.

The linear semigroup is stabilized:

\[ \forall t \geq 0, \quad \| e^{At} \|_{\mathcal{L}(X)} \leq Me^{-\lambda t} \]

Moreover, if \( f \) is of degree \( p \), then

\[ F : \begin{pmatrix} u \\ v \end{pmatrix} \in H^1_0(\Omega) \times L^2(\Omega) \mapsto \begin{pmatrix} 0 \\ f(x, u) \end{pmatrix} \in H^1_0(\Omega) \times L^2(\Omega) \] is compact.

\[ U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} F(U(s)) \, ds \]

\( \Rightarrow \) the bounded sets admits compact \( \omega \)-limit sets.
Step 3: a unique continuation property.

It is sufficient to show that the $\omega$–limit sets consists of equilibrium points. By Lasalle’s principle, the trajectories $U(t) = (u, \partial_t u)$ in the $\omega$–limit sets have constant energy. So we have

$$\partial_t \mathcal{E}(U(t)) = -\int_{\Omega} \gamma(x)|\partial_t u|^2 \, dx = 0.$$  

Since $\gamma(x) \geq \alpha > 0$, we have $\partial_t u \equiv 0$ and thus $u$ is an equilibrium point.
Asymptotic compactness ⇔ high frequencies are not really modified by the nonlinearity
Unique continuation ⇔ classify low-frequency solutions (equilibria, travelling fronts. . . )

Key arguments where we use $\gamma$ positive:

1. $\|e^{At}\|_{\mathcal{L}(X)} \leq Me^{-\lambda t}$ has finite integral on $[0, +\infty)$
2. if $\mathcal{E}(U(t))$ is constant, then $\int \gamma(x)|u_t|^2 = 0$ and $u(t)$ is constant.
A standard extension

1. A historic result

2. A standard extension

3. The disk with two holes

4. The disk with three holes

5. Conclusion
What happens when $\gamma(x)$ may vanish?
The decay of the linear semigroup

C. Bardos, G. Lebeau and J. Rauch (1992)

\[ \| e^{At} \|_{L^\infty(X)} \leq Me^{-\lambda t} \]

Any long enough geodesic meets the support of the damping \( \gamma \)
The unique continuation property

If $U(t)$ belongs to an $\omega$—limit set,

$$\partial_t \mathcal{E}(U(t)) = -\int_{\Omega} \gamma(x) |\partial_t u|^2 \, dx = 0.$$ 

So $v(t) = \partial_t u(t)$ vanishes in $\omega$ the support of $\gamma$. Thus, we have

$$v \equiv 0 \text{ in } \omega \times \mathbb{R} \quad \text{and} \quad \partial^{2}_{tt} v = \Delta v - f'(u(x, u(x, t))) v.$$ 

To conclude that $v \equiv 0$ everywhere, we need to use a unique continuation property. Basically, we may extend the zone where $v \equiv 0$ through convex surfaces.

[N. Lerner and L. Robbiano, 1985], [L. Hörmander, 1985], [Tataru, 1996]
The gradient dynamics hold for the domain with zero or one hole
A classic result

\[ \partial_{tt}^2 u + \gamma(x) \partial_t u = \Delta u - f(x, u) \]

Assume:

- \( \Omega \) is a two dimensional convex compact domain with or without a convex hole
- \( \gamma(x) \geq \alpha > 0 \) in a neighborhood of the exterior boundary of \( \Omega \).
- \( f \) of degree \( p \)
- \( f \) is asymptotically of the sign of \( u \)

Theorem

With the above assumptions, the dynamics of the damped wave equation admit a **compact global attractor** \( A \). Moreover, it is gradient-like and any trajectory converges to the sets of equilibrium points.
The disk with two holes

1. A historic result

2. A standard extension

3. The disk with two holes

4. The disk with three holes

5. Conclusion
Without the geometric control condition

In some cases, the geometric control condition does not hold, but very few geodesics miss the support of the damping.

\[ \| e^{At} U_0 \|_{H^1 \times L^2} \leq Me^{-\lambda \sqrt[3]{t}} \| U_0 \|_{H^2 \times H^1} \]

[N. Burq, 1993]
[N. Burq and M. Zworski, 2004]
[R.J. and C. Laurent, 2018]
The disk with two holes

\[ \partial_{tt}^2 u + \gamma(x) \partial_t u = \Delta u - f(x, u) \]

Assume:

- \( \Omega \) is a convex compact domain of dimension 2 with two convex holes
- \( \gamma(x) \geq \alpha > 0 \) in a neighborhood of the exterior boundary of \( \Omega \).
- \( f \) of degree \( p < \infty \)
- \( f \) is asymptotically of the sign of \( u \)


*With the above assumptions, the dynamics of the damped wave equation are gradient-like. Moreover, they admit a compact global (\( D(A), X \))–attractor \( A \) in the following sense: any bounded set of \( D(A) = (H^2 \cap H^1_0) \times H^1_0 \) is attracted by \( A \) in the norm \( X = H^1_0 \times L^2 \). Moreover, any trajectory converges to the sets of equilibrium points and the convergence is uniform in balls of \( D(A) \).*
The disk with two holes

Main arguments:

\[ \| e^{At} U_0 \|_{H^1 \times L^2} \leq M e^{-\lambda t^{1/3}} \| U_0 \|_{H^2 \times H^1} \]

\[ U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} F(U(s)) \, ds \]

+ some technical tricks
The disk with three holes

1. A historic result
2. A standard extension
3. The disk with two holes
4. The disk with three holes
5. Conclusion
The disk with three holes

If there are three holes or more, with additional technical assumptions, we still have the decay

\[ \| e^{At} U_0 \|_{H^1 \times L^2} \leq M e^{-\lambda \sqrt[3]{t}} \| U_0 \|_{H^2 \times H^1} \]

But the unique continuation property of Lerner-Robbiano-Hörmander-Tataru cannot be used:
An analytic unique continuation property


Assume $\omega \neq \emptyset$ and $v(t) = \partial_t u(t)$ solves

$$v \equiv 0 \text{ in } \omega \times \mathbb{R} \quad \text{and} \quad \partial_{tt}^2 v = \Delta v - f'_u(x, u(x, t)) v.$$ 

Assume moreover that $t \mapsto f'_u(x, u(x, t))$ is analytic then $v \equiv 0$ everywhere.

[J.K. Hale and G. Raugel, 2003] let us hope that if $f(x, u)$ is analytic in $u$, then a function $u$ in the attractor should be analytic in time and thus $f'_u(x, u(x, t))$ is also analytic.
An analytic unique continuation property

In the proofs of [J.K. Hale and G. Raugel, 2003], a global solution $u$ is split between the low-frequencies $P_n u$ and the high-frequencies $Q_n u$. It is used that

$$\| e^{At} U \|_X \leq M e^{-\lambda t} \| U \|_X \implies \| e^{Q_n A Q_n^t} Q_n U \|_X \leq N e^{-\mu t} \| Q_n U \|_X .$$

In our case, we would like to obtain

$$\| e^{At} U \|_X \leq M e^{-\lambda \sqrt[3]{t}} \| U \|_{D(A)} \implies \| e^{Q_n A Q_n^t} Q_n U \|_X \leq h(t) \| Q_n U \|_{D(A)} .$$

$\implies$ we adapt the ideas of [J.K. Hale and G. Raugel, 2003] but several technical problems have to be overcome.

[C.J.K. Batty and Th. Duyckaerts, 2008], [A. Borichev and Y. Tomilov, 2010],
[N. Anantharaman and M. Léautaud, 2014]
The disk with three holes

\[
\partial_{tt}^2 u + \gamma(x) \partial_t u = \Delta u - f(x, u)
\]

Assume:
- \( \Omega \) is as opposite and the holes are not aligned and small enough
- \( f(x, u) \) is analytic in \( u \)
- \( f \) of degree \( p < \infty \)
- \( f \) is asymptotically of the sign of \( u \)


*With the above assumptions, the dynamics admit a compact global \((D(A), X)\)–attractor. Moreover, any trajectory converges to the sets of equilibrium points and the convergence is uniform in balls of \( D(A) \).*
Conclusion

1. A historic result
2. A standard extension
3. The disk with two holes
4. The disk with three holes
5. Conclusion
Additional results

- **Other geometries are possible**

  The peanut of rotation

  \[ \| e^{At} U_0 \|_{H^1 \times L^2} \leq M e^{-\lambda \sqrt{t}} \| U_0 \|_{H^2 \times H^1} \]

  [E. Schenck, 2011]


  The torus with degenerated damping

  \[ \gamma(x) \sim |x_1|^\beta \]

  \[ \| e^{At} U_0 \|_{H^1 \times L^2} \leq \frac{C}{(1+t)^{1+2/\beta}} \| U_0 \|_{H^2 \times H^1} \]

  [M. Léautaud and N. Lerner, 2015]

- **Higher dimension**

  In dimension \( d = 3 \), assume that \( f \) is Sobolev-subcritical, that is of degree \( p \) with \( p < 3 \). It should also be possible to go to \( f \) energy-subcritical, that is of degree \( p \) with \( p < 5 \) by using Strichartz estimates, see [B. Dehman, G. Lebeau and E. Zuazua, 2003], [R.J. and C. Laurent, 2013]
Open problem

\[ U(t) = e^{At} U_0 + \int_0^t e^{A(t-s)} F(U(s)) \, ds \]

Open question n.1:

How important is the integrability of the linear decay?

For example, if the linear decay is simply

\[ \| e^{At} U_0 \|_{H^1 \times L^2} \leq \frac{C}{\ln(2 + t)} \| U_0 \|_{H^2 \times H^1} \]

does the asymptotic compactness hold?
Open question n.2:

In dispersive equations or if $F$ not compact, can we use appropriate estimates to adapt the proofs?
