On stability of N-solitons of a fourth order nonlinear Schrödinger equation

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Outline

Physical backgrounds

- Denote the centerline of the vortex filament by $X = X(t,x)$, (κ, τ) be the curvature and torsion, and $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ be the Frenet-Serret frame of centerline of the vortex filament, Da Rios [3] proposed vortex filament equation

$$\partial_t X = \kappa \mathbf{b} = \mathbf{X}' \times \mathbf{X}'', \tag{1.1}$$

- Hasimoto [5] found a connection between (1.1) and NLS

$$i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u = 0 \tag{1.2}$$

via the *Hasimoto transform*:

$$u(t,x) = \kappa(t,x) \exp(-i \int_0^t A(s) ds + i \int_0^x \tau(t,y) dy),$$

where

$$A(t) = -\frac{i\partial_t \kappa + \partial_x^2 \kappa + 2i\tau \partial_x \kappa + i\kappa \partial_x \tau - \kappa \tau^2 + \frac{1}{2}\kappa^3}{\kappa}\Big|_{x=0}.$$

- To describe the motion of an actual vortex filament precisely, Fukumoto and Moffatt [4] proposed the following higher order approximate equation

$$\partial_t X = \kappa \mathbf{b} - \nu \left(\kappa^2 \tau \mathbf{t} + (2\tau \partial_x \kappa + \kappa \partial_x \tau) \mathbf{n} + (\kappa \tau^2 - \partial_x^2 \kappa - \frac{1}{2}\kappa^3) \mathbf{b} \right). \tag{1.3}$$

- Using the Hasimoto transform again, we see that (1.3) is transformed to the following fourth order nonlinear Schrödinger type equation

$$\begin{aligned} i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u + \nu(\partial_x^4 u + \frac{3}{8}|u|^4 u + \frac{3}{2}(\partial_x u)^2 \bar{u} \\ + |\partial_x u|^2 u + \frac{1}{2} u^2 \partial_x^2 \bar{u} + 2|u|^2 \partial_x^2 u) = 0. \end{aligned} \tag{4NLS}$$

NLS

- When $\nu = 0$, (4NLS) reduces to the classical NLS

$$i\partial_t u + \partial_x^2 u + \frac{1}{2}|u|^2 u = 0. \tag{NLS}$$

- Solitons of (NLS), i.e. $u = e^{it}Q$ where $Q(x) = 2\,\mathrm{sech}(x)$ solves the following ODE:

$$-Q'' + Q - \frac{1}{2}Q^3 = 0. \tag{1.4}$$

- (NLS) has a number families of symmetries. Scaling, space-time transition, phase, Galilean transformation. So actually

$$R_0(t,x) = \sqrt{\omega}e^{i(\omega t - \frac{1}{4}c^2 t + \frac{1}{2}c\cdot x + \theta)}Q(\sqrt{\omega}(x - ct + \delta)) \tag{1.5}$$

is also a soliton solution of (NLS).

The following quantities are conserved formally along the flow of (4NLS)

$$
\begin{aligned}
H_0 &:= M(u) := \frac{1}{2}\int_{\mathbb{R}} |u|^2, \\
H_1 &:= P(u) := \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} u\partial_x \bar{u}, \\
H_2 &:= E(u) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 - \frac{1}{4}|u|^4, \\
H_3 &:= \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} \partial_x \bar{u}\partial_x^2 u - \frac{1}{2}|u|^2\mathrm{Re}(u\partial_x\bar{u}), \\
H_4 &:= F(u) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x^2 u|^2 + \frac{1}{2}\int_{\mathbb{R}} |u|^6 \mathrm{d}x \\
&+\frac{1}{4}\mathrm{Re}\int_{\mathbb{R}} (|u|^2\bar{u}\partial_x^2 u - 2|u|^2|\partial_x u|^2).
\end{aligned}
$$

For $m \geq 3$ the conservation quantities are as follows:

$$
G_m(u) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x^m u|^2 \mathrm{d}x + \int_{\mathbb{R}} q_m(u_R, u_I, ..., \partial_x^{m-1}u_R, \partial_x^{m-1}u_I)\mathrm{d}x,
$$

- (4NLS) is locally well-posed in H^s for $s \geq \frac{1}{2}$ (Huo and Jia [7]) and whereas ill-posed in H^s for $s < \frac{1}{2}$ (Maeda and Segata[9]). Global well-posed in H^m for $m \in \mathbb{Z}_+$.
- (4NLS) does not possess Galilean transform, which is one of the crucial differences between (NLS) and (4NLS).
- Hoseini and Marchant [6] found a two-parameter family of solitary waves (called Hasimoto soliton) of the form

$$R(t,x) := e^{i\beta t} Q_{\omega,\alpha}(x-ct) = e^{i\beta t} e^{i\alpha x} Q(\sqrt{\omega}(x-ct)), \tag{1.6}$$

where $\omega > 0,\ \alpha \in \mathbb{R},\ Q$ is defined in (1.4) and the above parameters satisfy the following relation:

$$\beta = \nu\alpha^4 + \nu\omega^2 - 6\nu\omega\alpha^2 - \alpha^2 + \omega, c = -4\nu\alpha^3 + 4\nu\alpha\omega + 2\alpha. \tag{1.7}$$

Stability of Hasimoto soliton

Theorem 1.1 (DCDS-A, 2017)

Let $\nu < 0$, $|\alpha|$ *large and* $\omega > 0$ *and let* β *and* c *be given by* (1.7). *The Hasimoto soliton* $e^{i\beta t}Q_{\omega,\alpha}(\cdot - ct)$ *defined by* (1.6) *is orbitally stable in* $H^2(\mathbb{R})$ *in the following sense: There exist parameters* ϵ_0, A_0 *such that the following holds. Consider* $u_0 \in H^2(\mathbb{R})$, *assume that* $\exists$ $\epsilon \in (0, \epsilon_0)$ *such that*

$$\|u_0 - Q_{\omega,\alpha}\|_{H^2(\mathbb{R})} < \epsilon, \tag{1.8}$$

then $\exists$ $\theta(t), y(t)$ *such that* $u(t) \in \mathcal{C}([0, +\infty), H^2(\mathbb{R}))$ *of* (4NLS), *with the initial data* $u(0) = u_0$, *satisfies*

$$\sup_{t\in(0,+\infty)} \|u(t) - e^{i\theta(t)+i\alpha x}Q_\omega(x - y(t))\|_{H^2(\mathbb{R})} < A_0\epsilon, \tag{1.9}$$

where

$$\sup_{t\in(0,+\infty)} |\theta'(t) - \beta| + |y'(t) - c| \leq CA_0\epsilon. \tag{1.10}$$

Corollary 1.2

The Hasimoto soliton $e^{i\beta t}Q_{\omega,\alpha}(x-ct)$ *is orbitally stable in* H^m *for* $m \in \mathbb{Z}_+$.

Remark 1.3

- $\nu = 0$, *Cazenave and Lions [1], orbitally stable in* H^1;
- $\nu \neq 0$, $\alpha = 0$, *Masaya and Segata 11 [9], orbitally stable in* H^m *for* $m \in \mathbb{Z}_+$
- $\nu \neq 0$, $\alpha \neq 0$, *Segata [13], two-parameter solitary wave* (1.6) *is orbitally stable in* H^1.
- $\nu < 0$, $|\alpha|$ *large, we proved the two-parameter soliton is orbitally stable in* H^2 *and in fact* H^m *for* $m \in \mathbb{Z}_+$.

Scheme of proof: Lyapunov functional

Hasimoto soliton R satisfies the following stationary equation

$$-\partial_x^2 R + \left(\beta + \alpha c\right) R - \frac{1}{2}|R|^2 R + ic\partial_x R$$
$$-\nu\left(\partial_x^4 R + \frac{3}{8}|R|^4 R + \frac{3}{2}(\partial_x R)^2 \bar{R} + |\partial_x R|^2 R + \frac{1}{2} R^2 \partial_x^2 \bar{R} + 2|R|^2 \partial_x^2 R\right) = 0.$$

Then R is a critical point of the following functional

$$S(u) := E(u) + (\beta + \alpha c) M(u) + cP(u) - \nu F(u). \tag{1.11}$$

i.e,

$$S'(R) = 0. \tag{1.12}$$

The Hessian of the action

$$H(\Upsilon) := \frac{1}{2}\langle S''(R)\Upsilon, \Upsilon\rangle. \tag{1.13}$$

then we have Taylor expansion

$$S(u(t)) = S(R(t)) + H(\Upsilon)(t) + O(\|\Upsilon(t)\|_{H^2}^3).$$

One main thing left to do is the coercivity of $H(\Upsilon)(t)$, we need to calculate the spectrum of the operator:

$$L_\alpha = \begin{pmatrix} L_{\alpha,+} & P \\ P^* & L_{\alpha,-} \end{pmatrix} \tag{1.14}$$

where

$$\begin{aligned} L_{\alpha,+} &= -(-6\nu\alpha^2+1)\partial_x^2 + (-\nu\alpha^4+\beta+\alpha^2) - \nu\partial_x^4 - (\frac{3}{2}-9\nu\alpha^2)Q^2 \\ -\quad & \nu(\frac{15}{8}Q^4 + \frac{5}{2}Q_x^2 + 5QQ_x\partial_x + 5QQ_{xx} + \frac{5}{2}Q^2\partial_x^2), \\ L_{\alpha,-} &= -(-6\nu\alpha^2+1)\partial_x^2 + (-\nu\alpha^4+\beta+\alpha^2) - \nu\partial_x^4 - (\frac{1}{2}-3\nu\alpha^2)Q^2 \\ -\quad & \nu(\frac{3}{8}Q^4 - \frac{1}{2}Q_x^2 + 3QQ_x\partial_x + QQ_{xx} + \frac{3}{2}Q^2\partial_x^2), \\ P &= -2\nu\alpha(2\omega\partial_x - 2\partial_x^3 - 3Q^2\partial_x), \\ P^* &= 2\nu\alpha(2\omega\partial_x - 2\partial_x^3 - 3Q^2\partial_x - 6QQ'). \end{aligned}$$

- Let us define two auxiliary linear operators as follows:

$$Mh(x) = h'(x) + \tanh(x)h(x), \quad M^t h = -h'(x) + \tanh(x)h(x). \quad (1.15)$$

M and M^t map odd functions in even functions and even functions in odd functions. Moreover, the null space of M is spanned by Q and M^t is injective; M is onto and the image of M^t is the subspace orthogonal to Q.

- Recall that $Q(x) = 2\sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}x)$. The operator

$$S''(Q)\varphi = -\partial_x^2\varphi + c\varphi - \frac{3}{2}Q^2\varphi$$
$$-\nu\left(\partial_x^4\varphi + \frac{15}{8}Q^4\varphi + \frac{5}{2}Q_x^2\varphi + 5QQ_x\varphi_x + 5QQ_{xx}\varphi + \frac{5}{2}Q^2\varphi_{xx}\right).$$

is a compact perturbation of the constant coefficient operator

$$\tilde{L} = -\nu\partial_x^4 - \partial_x^2 + c, \quad c = \omega(1+\nu\omega) > 0,$$

- $S''(Q)$ satisfies the following operator identity:

$$MS''(Q)M^t = M^t\tilde{L}M. \tag{1.16}$$

  We can employ the spectral analysis of $S''(Q)$ in H^2_{odd} and H^2_{even}.
- The above approach appears in [12] which study the isoinertial family of operators, this technique was used to prove stability of multi-solitons of KdV equation (Maddocks and Sachs 93 [8]) and BO equation (Neves and Lopes 06 [12]).
- The above identity is useful in proving the stability of multi-solitons of mKdV equation.

Theorem 1.4

Suppose that $\nu < 0$ *and* $|\alpha| >> 1$ *such that* $-\nu\alpha^4 + \beta + \alpha^2 > 0$*. The operator* L_α *defined on* $L^2(\mathbb{R})$ *with domain* $H^4(\mathbb{R})$ *has a unique negative eigenvalue, which is simple. The eigenvalue zero is double with associated eigenfunctions* Q' *and* iQ*. Moreover, the essential spectrum is the interval* $[-\nu\alpha^4 + \beta + \alpha^2, \infty)$*.*

Theorem 1.5

Assume $|\alpha|$ *is large. Let* R *be given by* (1.6)*, and* H *the functional defined by* (1.13)*. There exists* $\lambda > 0$ *such that for any* $\Upsilon \in H^2(\mathbb{R})$ *satisfying the following orthogonality conditions*

$$(\Upsilon, R)_{L^2} = (\Upsilon, R')_{L^2} = (\Upsilon, iR)_{L^2} = 0, \tag{1.17}$$

then we have

$$H(\Upsilon) \geq \lambda \|\Upsilon\|^2_{H^2}.$$

Define

$$R_j(t,x) := 2e^{i(\beta_j t+\alpha_j x+\tau_j)} Q(\sqrt{\omega_j}(x-c_j t-x_j)). \tag{2.1}$$

Theorem 2.1

Let $\nu < 0$. For $j = 1, ..., N$, let $|\alpha_j|$ large and $(\omega_j, \beta_j, \tau_j, x_j, c_j) \in \mathbb{R}^+ \times \mathbb{R}^4$ be sets of parameters which satisfy (1.7) and R_j be defined as (2.1). Define

$$c_* := \frac{1}{4}\min|c_j - c_k|; \qquad j,k = 1, ..., N, j \neq k, \qquad \omega_0 := \frac{1}{4}\min\{\omega_j\}.$$

then there exists solution $u(t)$ of (4NLS), such that for some $T_0 > 0$ and all $t \in [T_0, +\infty)$ the following estimate holds:

$$\|u(t) - \sum_{j=1}^{N} R_j(t)\|_{H^2} \leq e^{-\sqrt{\omega_0}c_* t},$$

Remark 2.2

No work concerns the IST theory of (4NLS) *in current literature, notice that we do not use the information of its integrability. To our knowledge, this is the first time that such solutions are exhibited for fourth order type Schrödinger equations.*

- The new ingredients of the proof are modulation theory, energy method, virial identity adapted to 4NLS.

Scheme of proof : Backward resolution of (4NLS)

Take time sequence $(T^n) \uparrow +\infty$ and u_n solutions of (4NLS) with final data $u_n(T^n) = R(T^n)$.

- Approximate multi solitary waves. Show that for each n, u_n exists on $[T_0, T^n]$ with T_0 independent of n.
- Convergence. Show that u_n converges to a multi solitary waves.

The major tools

- Uniform estimates. Coercivity of the Hessian of the action around the multi solitary waves, slow variation of almost conservation laws.
- Compactness argument.

Proposition 2.3 (Uniform estimates)

There exist $T_0 \in \mathbb{R}$ *(independent of* n*) such that for* n *large enough the solution* u_n *of* (4NLS) *with* $u_n(T^n) = \sum_{j=1}^{N} R_j(T^n)$ *exists on* $[T_0, T^n]$ *and satisfies for all* $t \in [T_0, T^n]$ *the estimate*

$$\|u_n(t) - \sum_{j=1}^{N} R_j(t)\|_{H^2} \leq e^{-\sqrt{\omega_0} c_* t} \tag{2.2}$$

Proposition 2.4 (Compactness argument)

Let T_0 *be given by Proposition 2.3. There exists* $u_0 \in H^2$ *such that, possibly replaced with a subsequence,* $u_n(T_0) \to u_0$ *strongly in* H^s *for any* $s \in [0, 2)$ *as* $n \to +\infty$.

In fact u_0 is the initial data to born multi solitary waves.

Proof of Theorem 2.1

Suppose uniform estimate and compactness argument hold, $u(t)$ is a solution of (4NLS) with initial data $u(T_0) = u_0$, for $t > T_0$ and $s \in [0, 2)$, we have

$$u_n(t) \to u(t) \quad \text{in} \quad H^s(\mathbb{R});$$

$$u_n(t) \rightharpoonup u(t) \quad \text{in} \quad H^2(\mathbb{R}).$$

which indicates that for $t \in [T_0, T_\infty)$,

$$\|u(t) - R(t)\|_{H^2} \leq \liminf_{n\to\infty} \|u_n(t) - R(t)\|_{H^2} \leq e^{-\sqrt{\omega_0}c_* t},$$

therefore, $u(t)$ is a multi solitary wave of (4NLS).

Proof of Uniform estimate

Proposition 2.5 (Bootstrap)

There exist $T_0 \in \mathbb{R}$ *(independent of* n*) such that for* n *large enough the following bootstrap property holds: For* $t_0 \in [T_0, T^n]$ *and all* $t \in [t_0, T^n]$*, if* u_n *satisfies the following estimate*

$$\|u_n(t) - R(t)\|_{H^2} \le e^{-\sqrt{\omega_0}c_\star t}, \tag{Bootstrap-1}$$

then for all $t \in [t_0, T^n]$*, it will also satisfies the following better estimate*

$$\|u_n(t) - R(t)\|_{H^2} \le \frac{1}{2} e^{-\sqrt{\omega_0}c_\star t}. \tag{Bootstrap-1/2}$$

Modulation theory

For given (ϵ, L), we consider a neighborhood of the sum of Hasimoto solitons

$$\mathcal{U}(\epsilon, L) := \left\{ u \in H^2; \inf_{\substack{\xi_j > \xi_{j-1}+L \\ \vartheta_j \in \mathbb{R} \\ j=1,\ldots,N}} \|u - \sum_{j=1}^{N} e^{i\vartheta_j} Q_{\omega_j,\alpha_j}(\cdot - \xi_j)\|_{H^2} < \epsilon \right\}$$

Proposition 2.6

There exists $\tilde{\epsilon},\ \tilde{L},\ C,\ \tilde{C} > 0$ *such that for any* $0 < \epsilon < \tilde{\epsilon}$ *and* $L > \tilde{L}$ *the following property is verified.*
Let $u(t,x)$ *be a solution of* (4NLS) *satisfying on a time interval* I *,*

$$u \in \mathcal{U}(\epsilon, L), \qquad \text{for all } t \in I.$$

For $j = 1, \ldots, N$, *there exist (unique)* $\mathcal{C}^1$ *functions*

$$\tilde{\theta}_j : I \to \mathbb{R}, \qquad \tilde{\omega}_j : I \to \mathbb{R}^+, \qquad \tilde{x}_j : I \to \mathbb{R},$$

$$\tilde{R}_j(t) = e^{i\tilde{\theta}_j(t)+i\alpha_j x} Q_{\tilde{\omega}_j(t)}(x - \tilde{x}_j(t)), \ \Upsilon(t) = u(t) - \sum_{j=1}^{N} \tilde{R}_j(t),$$

then Υ satisfies for all $t \in I$ the orthogonality conditions

$$(\Upsilon, i\tilde{R}_j)_{L^2} = (\Upsilon, \tilde{R}_j)_{L^2} = (\Upsilon, \partial_x \tilde{R}_j)_{L^2} = 0, \quad j = 1, \ldots, N.$$

Moreover, for all $t \in I$ we have

$$\|\Upsilon\|_{H^2} + \sum_{j=1}^{N} |\tilde{\omega}_j - \omega_j| \leq \tilde{C}\epsilon, \qquad \tilde{x}_{j+1} - \tilde{x}_j > \frac{L}{2}, \ j = 1, \ldots, N-1,$$

and the derivatives in time verify

$$\sum_{j=1}^{N} \left(|\partial_t \tilde{\omega}_j| + \left|\partial_t \tilde{\theta}_j - \tilde{\beta}_j\right|^2 + |\partial_t \tilde{x}_j - \tilde{c}_j|^2 \right) < C \left(\|\Upsilon\|_{H^2}^2 + e^{-3\sqrt{\omega_0} c_* t} \right).$$

Localization procedure

For $j = 1, ..., N$, set

$$E_j(u) := \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 \phi_j \mathrm{d}x - \frac{1}{8}\int_{\mathbb{R}} |u|^4 \phi_j \mathrm{d}x,$$

$$F_j(u) :== \int_{\mathbb{R}} \frac{1}{2}|\partial_x^2 u|^2 \phi_j + \frac{1}{16}|u|^6 \phi_j + \frac{1}{4}\mathrm{Re}(|u|^2 \bar{u}\partial_x^2 u)\phi_j - \frac{1}{2}|u|^2|\partial_x u|^2 \phi_j \mathrm{d}x,$$

$$M_j(t,u) := \frac{1}{2}\int_{\mathbb{R}} |u|^2 \phi_j \mathrm{d}x, \qquad P_j(t,u) := \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} u\partial_x \bar{u}\phi_j \mathrm{d}x.$$

$$S_j(t,u) := E_j(u) + (\beta_j + \alpha_j)\, M_j(t,u) + c_j P_j(t,u) - \nu F_j(t,u)$$

$$\mathcal{H}(t,\varepsilon(t)) := \sum_{j=1}^{N} \langle S_j''(\tilde{R}_j(t))\varepsilon(t), \varepsilon(t)\rangle.$$

Proposition 2.7 (Coercivity)

There exists $\lambda > 0$ *such that for* $t \in [t_0, T^n]$, *we have*

$$\mathcal{H}(t,\varepsilon) \geq \lambda \|\varepsilon\|_{H^2}^2.$$

Proposition 2.8 (Almost conservation law)

For $t \in [t_0, T^n]$, *we have*

$$\left|\frac{\partial}{\partial t}\mathcal{S}(t, u_n(t))\right| \leq \frac{C(c_1, ..., c_N)}{\sqrt{t}} e^{-2\sqrt{\omega_0}c_* t}.$$

Proof of bootstrap

$$\|u_n - R\|_{H^2}^2 \leq 2\|\tilde{R} - R\|_{H^2}^2 + 2\|\varepsilon\|_{H^2}^2.$$

$$\|\tilde{R}-R\|_{H^2}^2 \leq C\left(\sum_{j=1}^{N} |\tilde{\omega}_j(t)-\omega_j|^2+|\tilde{\theta}_j(t)-\theta_j|^2+|\tilde{x}_j(t)-ct-x_j|^2\right) \leq o\left(e^{-2\sqrt{\omega_0}c_\star t}\right).$$

$$\lambda\|\varepsilon\|_{H^2}^2 \leq \mathcal{H}(t,\varepsilon(t)) = \mathcal{S}(t,u_n(t))-\mathcal{S}(T_n,u_n(T_n))+o(e^{-2\sqrt{\omega_0}c_* t}) \leq o(e^{-2\sqrt{\omega_0}c_* t}),$$

Choose $t > T_0$, T_0 is independent of n,

$$\|u_n - R\|_{H^2} \leq o(e^{-\sqrt{\omega_0}c_\star t}) \leq \frac{1}{2}e^{-\sqrt{\omega_0}c_\star t}.$$

Which finishes the Bootstrap argument.

Review of the proof

- Backward resolution of (4NLS)
- Uniform estimates
  - Modulation theory
  - Localization procedure
  - Energy control
- Compactness argument for the initial data
  - Virial identity

- 4NLS has N-solitons, that are given by explicit formula. These special solutions can be regarded as the nonlinear superposition of N single solitons which interact nontrivially and then separate.
- Let $\delta = (\delta_1, \delta_2, \cdots, \delta_N) \in \mathbb{R}^N$, $\theta = (\theta_1, \theta_2, \cdots, \theta_N) \in \mathbb{R}^N$ and $(\alpha, \omega) \in S_N$, where

$$S_N = \{(\alpha, \omega); \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N) \in \mathbb{R}^N, \omega = (\omega_1, \omega_2, \cdots, \omega_N) \in \mathbb{R}_+^N, \alpha_i \neq \alpha_j \text{ for } 1 \leq i < j \leq N.\}$$

- N-solitons $U^{(N)}(\omega, \theta, \alpha, \delta; t, x)$. Define

$$G_{\alpha,\omega} = \{u \in H^N; H_k(u) = H_k(U^{(N)}(\omega, \theta, \alpha, \delta; t, x)) \text{ for } 2 \leq k \leq 2N.\}$$
$$M_{\alpha,\omega} = \{u \in H^N; u = U^{(N)}(\omega, \theta, \alpha, \delta; t, x) \text{ for some } \theta, \delta \in \mathbb{R}^N.\}$$

- $G_{\alpha,\omega}$ is independent of θ, δ. $M_{\alpha,\omega} \subseteq G_{\alpha,\omega}$, in particular, when $N=1$, $M_{\alpha,\omega} = G_{\alpha,\omega}$; for $N>1$, the question $M_{\alpha,\omega} = G_{\alpha,\omega}$ appears to be open, even for KdV equation.

Theorem 3.1 (Stability of multi-solitons)

Suppose $(\alpha,\omega) \in S_N$. For every $\epsilon > 0$, there exists $A > 0$ such that if $u_0 \in H^N$, $\theta_0, \delta_0 \in \mathbb{R}^N$ and $\|u_0 - U^N(\omega, \theta_0, \alpha, \delta_0; 0, x)\|_{H^N} < A\epsilon$, then for all $t > 0$,

$$\inf_{\psi \in G_{\alpha,\omega}} \|u(t) - \psi(t)\|_{H^N} < \epsilon.$$

Remark 3.2

Dynamical stability of multi-solitons of completely integrable systems like KdV hierarchy, BO and higher order BO, NLS and three-wave, have been proved. Kapitula [14] proved the NLS case by the method of IST.

- In view of Hasimoto soliton profile $R(t,x) = U^{(1)}(\omega,\theta,\alpha,\delta;t,x)) = e^{i\beta t+\theta}e^{i\alpha x}Q(\sqrt{\omega}(x-ct+\delta))$, it is a critical point of the functional

$$H_2(u) + 2\alpha H_1(u) + (\omega+\alpha^2)H_0(u)$$

.

- For 2-soliton $U^{(2)}(\omega,\theta,\alpha,\delta;t,x)$, it is a critical point of the functional

$$I(u;\omega,\alpha) := H_4(u) + \lambda_1 H_3(u) + \lambda_2 H_2(u) + \lambda_3 H_1(u) + \lambda_4 H_0(u),$$

where

$$\begin{aligned}
\lambda_1 &= 2(\alpha_1+\alpha_2),\\
\lambda_2 &= (\omega_1+\omega_2+4\alpha_1\alpha_2+\alpha_1^2+\alpha_2^2),\\
\lambda_3 &= \left[2\alpha_2(\omega_1+\alpha_1^2)+2\alpha_1(\omega_2+\alpha_2^2)\right],\\
\lambda_4 &= (\omega_1+\alpha_1^2)(\omega_2+\alpha_2^2).
\end{aligned}$$

.

- Define

$$L = I''(U^{(2)};\omega,\alpha) := H_4''(U^{(2)}) + \lambda_1 H_3(U^{(2)}) + \lambda_2 H_2''(U^{(2)}) + \lambda_3 H_1''(U^{(2)}) + \lambda_4 H_0''(U^{(2)}),$$

 Denote $n(L)$ the number of negative eigenvalue of the self-ajoint operator L.

- The operator L is elliptic in the sense that it has a finite number of negative eigenvalues, a finite dimensional null-space, and the continuous spectrum is bounded away from zero from below. Since the principle part of L is

$$\begin{aligned}\tilde{L} &= \partial_x^4 - i\lambda_1\partial_x^3 - \lambda_2\partial_x^2 + i\lambda_3\partial_x + \lambda_4 \\ &= (-\partial_x^2 + 2\alpha_1 i\partial_x + \alpha_1^2 + \omega_1)(-\partial_x^2 + 2\alpha_2 i\partial_x + \alpha_2^2 + \omega_2).\end{aligned}$$

- Define the Hessian matrix of I

$$D := \left\{ \frac{\partial^2 I(U^{(2)})}{\partial\lambda_i \partial\lambda_j} \right\},$$

Denote $p(D)$ the number of positive eigenvalue of the matrix D.

-

$$H'_{n+2}(U^{(1)}) + 2\alpha H'_{n+1}(U^{(1)}) + (\omega + \alpha^2) H'_n(U^{(1)}) = 0, \ n \geq 0,$$

which implies the following equations

$$\frac{\partial}{\partial\alpha} H_{n+2}(U^{(1)}) + 2\alpha \frac{\partial}{\partial\alpha} H'_{n+1}(U^{(1)}) + (\omega + \alpha^2) \frac{\partial}{\partial\alpha} H'_n(U^{(1)}) = 0,$$
$$\frac{\partial}{\partial\omega} H_{n+2}(U^{(1)}) + 2\alpha \frac{\partial}{\partial\omega} H'_{n+1}(U^{(1)}) + (\omega + \alpha^2) \frac{\partial}{\partial\omega} H'_n(U^{(1)}) = 0$$

Quantitive property of 1-soliton

- Recall that

$$H_0(U^{(1)}) = 4\sqrt{\omega},$$
$$H_1(U^{(1)}) = -4\alpha\sqrt{\omega}.$$

- We can calculate from above to obtain

$$H_2(U^{(1)}) = 4\alpha^2\sqrt{\omega} - \frac{4}{3}\omega^{\frac{3}{2}},$$
$$H_3(U^{(1)}) = -4\alpha^3\sqrt{\omega} + 4\alpha\omega^{\frac{3}{2}},$$
$$H_4(U^{(1)}) = 4\alpha^4\sqrt{\omega} - 8\alpha^2\omega^{\frac{3}{2}} + \frac{4}{5}\omega^{\frac{5}{2}}.$$

- For 2-soliton $U^{(2)}$

$$r_0 := H_0(U^{(2)}) = 4\sum_{j=1}^{2}\sqrt{\omega_j},$$

$$r_1 := H_1(U^{(2)}) = -4\sum_{j=1}^{2}\alpha_j\sqrt{\omega_j},$$

$$r_2 := H_2(U^{(2)}) = 4\sum_{j=1}^{2}\alpha_j^2\sqrt{\omega_j} - \frac{4}{3}\sum_{j=1}^{2}\omega_j^{\frac{3}{2}},$$

$$r_3 := H_3(U^{(2)}) = -4\sum_{j=1}^{2}\alpha_j^3\sqrt{\omega_j} + 4\sum_{j=1}^{2}\alpha_j\omega_j^{\frac{3}{2}},$$

$$r_4 := H_4(U^{(2)}) = 4\sum_{j=1}^{2}\alpha_j^4\sqrt{\omega_j} - 8\sum_{j=1}^{2}\alpha_j^2\omega_j^{\frac{3}{2}} + \frac{4}{5}\sum_{j=1}^{2}\omega_j^{\frac{5}{2}}.$$

Theorem 3.1 ($N = 2$) follows from

Theorem 3.3

Suppose that $U^{(2)}$ is a non-degenerate constrained minimum of

$$\min H_4(u), \quad \textit{subject to } H_j(u) = r_j,\ j = 0, 1, 2, 3.$$

and

$$n(L) = p(D). \tag{3.1}$$

Then there exists a $C > 0$ such that $U^{(2)}$ is a non-degenerate unconstrained minimum of the augmented Lagrangian (Lyapunov function)

$$I(u) + \frac{C}{2}\sum_{j=0}^{3}(H_j(u) - r_j)^2. \tag{3.2}$$

By the definition

$$D := MQ^{-1} = \begin{pmatrix} \frac{\partial H_3}{\partial \omega_1} & \frac{\partial H_3}{\partial \alpha_1} & \frac{\partial H_3}{\partial \omega_2} & \frac{\partial H_3}{\partial \alpha_2} \\ \frac{\partial H_2}{\partial \omega_1} & \frac{\partial H_2}{\partial \alpha_1} & \frac{\partial H_2}{\partial \omega_2} & \frac{\partial H_2}{\partial \alpha_2} \\ \frac{\partial H_1}{\partial \omega_1} & \frac{\partial H_1}{\partial \alpha_1} & \frac{\partial H_1}{\partial \omega_2} & \frac{\partial H_1}{\partial \alpha_2} \\ \frac{\partial H_0}{\partial \omega_1} & \frac{\partial H_0}{\partial \alpha_1} & \frac{\partial H_0}{\partial \omega_2} & \frac{\partial H_0}{\partial \alpha_2} \end{pmatrix} \begin{pmatrix} \frac{\partial \lambda_1}{\partial \omega_1} & \frac{\partial \lambda_1}{\partial \alpha_1} & \frac{\partial \lambda_1}{\partial \omega_2} & \frac{\partial \lambda_1}{\partial \alpha_2} \\ \frac{\partial \lambda_2}{\partial \omega_1} & \frac{\partial \lambda_2}{\partial \alpha_1} & \frac{\partial \lambda_2}{\partial \omega_2} & \frac{\partial \lambda_2}{\partial \alpha_2} \\ \frac{\partial \lambda_3}{\partial \omega_1} & \frac{\partial \lambda_3}{\partial \alpha_1} & \frac{\partial \lambda_4}{\partial \omega_2} & \frac{\partial \lambda_4}{\partial \alpha_2} \\ \frac{\partial \lambda_4}{\partial \omega_1} & \frac{\partial \lambda_4}{\partial \alpha_1} & \frac{\partial \lambda_4}{\partial \omega_2} & \frac{\partial \lambda_4}{\partial \alpha_2} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{2\alpha_1^3}{\sqrt{\omega_1}} + 6\alpha_1\sqrt{\omega_1} & -12\alpha_1^2\sqrt{\omega_1} + 4\omega_1^{\frac{3}{2}} & -\frac{2\alpha_2^3}{\sqrt{\omega_2}} + 6\alpha_2\sqrt{\omega_2} & -12\alpha_2^2\sqrt{\omega_2} \\ \frac{2\alpha_1^2}{\sqrt{\omega_1}} - 2\sqrt{\omega_1} & 8\alpha_1\sqrt{\omega_1} & \frac{2\alpha_1^2}{\sqrt{\omega_2}} - 2\sqrt{\omega_2} & 8\alpha_2\sqrt{\omega_2} \\ -\frac{2\alpha_1}{\sqrt{\omega_1}} & -4\sqrt{\omega_1} & -\frac{2\alpha_2}{\sqrt{\omega_2}} & -4\sqrt{\omega_2} \\ -2\sqrt{\omega_1} & 0 & -2\sqrt{\omega_2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 1 & 2\alpha_1 + 4\alpha_2 & 1 & 4\alpha_1 + 2\alpha_2 \\ 2\alpha_2 & 2(\omega_2 + 2\alpha_1\alpha_2 + \alpha_2^2) & 2\alpha_1 & 2(\omega_1 + \alpha_1^2 + 2\alpha_1\alpha_2) \\ \omega_2 + \alpha_2^2 & 2\alpha_1(\omega_2 + \alpha_2^2) & \omega_1 + \alpha_1^2 & 2(\omega_1 + \alpha_1^2)\alpha_2 \end{pmatrix}^{-1}.$$

Since $Q^TDQ = Q^TM$, by Sylvester's law of inertia, to find the number of positive eigenvalues of M it suffices to consider the number of positive eigenvalues of the matrix Q^TM.

$$Q^TM = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

$$A = \begin{pmatrix} \frac{2\left(\omega_2-\omega_1+(\alpha_1-\alpha_2)^2\right)}{\sqrt{\omega_1}} & 8(\alpha_1-\alpha_2)\sqrt{\omega_1} \\ 8(\alpha_1-\alpha_2)\sqrt{\omega_1} & -8\sqrt{\omega_1}\left(\omega_2-\omega_1+(\alpha_1-\alpha_2)^2\right) \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{2\left(\omega_1-\omega_2+(\alpha_1-\alpha_2)^2\right)}{\sqrt{\omega_2}} & 8(\alpha_2-\alpha_1)\sqrt{\omega_2} \\ 8(\alpha_2-\alpha_1)\sqrt{\omega_2} & -8\sqrt{\omega_2}\left(\omega_1-\omega_2+(\alpha_1-\alpha_2)^2\right) \end{pmatrix}.$$

Q^TM posses two positive eigenvalues, $P(D) = 2$.

- Since

$$U^{(2)}(\omega,\theta,\alpha,\delta;t,x) \to \sum_{j=1}^{2} U^{(1)}(\omega_j,\theta_j,\alpha_j,\delta_j;t,x), \text{ as } t \to +\infty.$$

Using the idea of Neves and Lopes [12], the study of the spectrum of $L = I''(U^{(2)})$ reduces to consider the following two operators,

$$L_j := I''(U^{(1)}(\omega_j,\theta_j,\alpha_j,\delta_j;t,x)),\ j = 1,2.$$

We can show that L_j has a unique negative eigenvalue, which is simple. The eigenvalue zero is double with associated eigenfunctions $\partial_x U^{(1)}(\omega_j,\theta_j,\alpha_j,\delta_j;t,x)$ and $iU^{(1)}(\omega_j,\theta_j,\alpha_j,\delta_j;t,x)$. Which is similar to analysis we presented before.

- Therefore, $L = I''(U^{(2)})$ have two negative eigenvalues, $n(L) = 2$.

Open problems

- Degenerate cases like $\omega_1 = \omega_2, \alpha_1 = \alpha_2$, double-pole solution, stability or not.
- For Schrödinger equation. Oribital stability, uniqueness and asymptotic stability of multi-solitons in H^1.
- Other integrable or non-integrable dispersive equations, KP equation, Davey-Stewartson system (DSI, DSII, Lump solution), ILW, etc.

References I

[1] T. Cazenave and P. L. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, *Commun. Math. Phys.*, **85**, 549–561 (1982).

[2] R. Côte, Y. Martel and F. Merle, Construction of multi-soliton solutions for the L^2-supercritical gKdV and NLS equations. *Rev. Mat. Iberoam.*, 27(1):273–302, 2011.

[3] L. S. Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, Rend. Circ. Mat. Palermo **22**, 117–135 (1906) (in Italian).

[4] Y. Fukumoto and H. K. Moffatt, Motion and expansion of a viscous vortex ring. Part I. A higher-order asymptotic formula for the velocity, J. Fluid Mech. **417**, 1–45 (2000).

[5] H. Hasimoto, A soliton on a vortex filament, J. Fluid Mech. **51**, 477–485 (1972).

[6] S. M. Hoseini and T. R. Marchant, Solitary wave interaction for a higher-order nonlinear Schrödinger equation, IMA J. Appl. Math. 72, 206–222 (2007).

[7] Z. Huo and Y. Jia, A refined well-posedness for the fourth-order nonlinear Schrödinger equation related to the vortex filament, Comm. Partial Differential Equations **32** (2007), no. 10–12, 1493–1510.

[8] J. H. Maddocks and R. L. Sachs, On the stability of KdV multi-solitons, *Comm. Pure Appl. Math.*, **46** (1993), 867–901.

[9] M. Maeda and J. Segata, Existence and stability of standing waves of fourth order nonlinear Schrödinger type equation related to vortex filament. Funkcial. Ekvac. **54** (2011), no. 1, 1–14.

References II

[10] M. Maeda and J. Segata, Existence and stability of standing waves of fourth order nonlinear Schrödinger type equation related to vortex filament. Funkcial. Ekvac. **54** (2011), no. 1, 1–14.

[11] Y. Martel and F. Merle, Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(6):849–864, 2006.

[12] A. Neves and O. Lopes, Orbital stability of double solitons for the Benjamin-Ono equation, *Comm. Math. Phys.*, **262** (2006), 757–791.

[13] J. Segata, Orbital stability of a two parameter family of solitary waves for a fourth order nonlinear Schrödinger type equation. J. Math. Phys. **54** (2013), no. 6, 061503, 6 pp.

[14] T. Kapitula, On the stability of N-solitons in integrable systems. Nonlinearity **20** (2007) 879 – 907

Thank You For Your Attention